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# The consumer's index

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**Abstract**

Consumer's surplus measures the area under the demand curve between two prices, but is path dependent. There exists a path such that consumer's surplus tracks utility and an explicit formula is known for CES utilities. This paper shows that the CES-based formula holds for any homothetic utility, and I call it the consumer's index. The index modifies consumer's surplus in two ways: the change in income is measured by its growth factor and the area under the demand curve is normalized by income.

**KEYWORDS**

consumer's surplus, price–income indices, purchasing power

**JEL CLASSIFICATION**

C43; D60

## 1 | INTRODUCTION

The measurement of purchasing power is a key issue. Purchasing power is an index of price $p$, an $n$-dimensional non-negative row vector, and income $m$, a positive scalar. An index is economic if it is based on the utility level of consumers. A consumer is supposed to solve the problem

$$\max U(x): \; px \leq m, \tag{1}$$

where $x$ is the consumption vector and $U$ the utility function. The solution to the consumer's problem (1) is demand $x = D(p, m)$, with value $V(p, m) = U(D(p, m))$. $V$ is called the indirect utility function. An index, $I(p, m)$, *tracks utility* if it preserves the order of $V(p, m)$: $I(p^1, m^1) \geq I(p^0, m^0)$ if and only if $V(p^1, m^1) \geq V(p^0, m^0)$. An index cannot be (indirect) utility itself, because that is not observed. Indices are based on demand, which is observable.

The three main indices for price changes are equivalent variation $EV$, compensating variation $CV$, and consumer's surplus $CS$. The equivalent variation of just a price change, from $p^0$ to $p^1$ (while $m^0 = m^1 = m$), is the equivalent reduction of the budget, yielding the new utility level under the old price: $V(p^0, m - EV) = V(p^1, m)$. The compensating variation of the price change is the increase of the budget, yielding the old utility level under the new price: $V(p^1, m + CV) = V(p^1, m)$. The equivalent and the compensating variations are indirectly observable, by letting the consumer choose between the old and the new price regime and adjusting the budget in one of the two regimes up to the point where the consumer switches. More directly observable is consumer's surplus, $\int_{p^0}^{p^1} D(p, m)dp$. The three indices are increasing in the price change. They are transformed to purchasing power indices by inclusion of a minus sign and addition of the income change, $m^1 - m^0$. Thus, the consumer's surplus index is defined by

$$CS = m^1 - m^0 - \int_{(p^0,m^0)}^{(p^1,m^1)} D(p, m)dp. \tag{2}$$

Willig (1976) calls consumer's surplus an "approximation" to the "exact" measures of equivalent and compensating variation, but Takayama (1982) argues that consumer's surplus is the better foundation for an index. I side with Takayama, though consumer's surplus is problematic, but two modifications can straighten this out, at least when utility is homothetic. The essence of homotheticity is that demand is linear in income: $D(p, m) = mD(p, 1)$. In the remainder of this paper I defend my siding with Takayama and discuss the issue of consumer's surplus. This will lead me to define a close cousin of consumer's surplus: the consumer's index.

## 2 | HOMOTHETIC UTILITY

Utility is *homothetic* when it is an increasing transformation of a homogeneous function of positive degree. Without loss of generality, by taking an increasing transformation of the utility function, ten Raa (2017) shows that the indirect utility that comes with homothetic utility fulfills

$$V(p, m) = V(p, 1) + \ln m, \tag{3}$$

and that, by Roy's lemma,

$$\frac{\partial V(p, m)}{\partial p} = -D(p, m)\frac{\partial V(p, m)}{\partial m} = -\frac{D(p, m)}{m},$$

we have

$$V(p^1, m^1) - V(p^0, m^0) = \ln m^1 - \ln m^0 - \int_{(p^0,m^0)}^{(p^1,m^1)} \left[\frac{D(p, m)}{m}\right]dp. \tag{4}$$

In Equation (4) the leading terms stem from the last term of Equation (3), while the last term stems from the first term of Equation (3), $V(p, 1)$, which is independent of $m$. Indeed, $D(p,m)/m$ is also independent of $m$ for homothetic utility.

## 3 | THE CONSUMER'S INDEX

I define the consumer's index by

$$CI = \ln m^1 - \ln m^0 - \int_{(p^0,m^0)}^{(p^1,m^1)} \left[\frac{D(p, m)}{m}\right]dp. \tag{5}$$

Obviously, definition (5) is motivated by Equation (4), but, at least in principle, it is also applicable to non-homothetic utilities. For homothetic utility, Equation (4) shows that the consumer's index tracks utility.

The consumer's index (5) is reminiscent of consumer's surplus (2), but there are two modifications. The consumer's index measures the change in income by the logarithmic growth factor and it normalizes the area under the demand curve (or surface) by income. The latter modification resolves a well-known flaw of consumer's surplus, at least for homothetic utility, namely the path dependence of the line integral in Equation (2). This path dependence even plagues the consumer's surplus associated with homothetic utility. Indeed, in that case (2) reduces to

$$CS = m^1 - m^0 - \int_{(p^0,m^0)}^{(p^1,m^1)} D(p,1)m dp, \tag{6}$$

and to illustrate the path dependence, take a path $(p^0,m^0) \rightarrow (p^0,m) \rightarrow (p^1,m) \rightarrow (p^1,m^1)$. I call such a path rectangular. The rectangularity merely concerns the division between price and income; the price path, $p^0 \rightarrow p^1$, need not be rectangular with respect to price components. For a rectangular path Equation (6) reduces to

$$CS = m^1 - m^0 - m\int_{p^0}^{p^1} D(p,1)dp. \tag{7}$$

Equation (7) illustrates the path dependency of consumer's surplus, simply because the path parameter $m$ may vary. In contrast, the consumer's index is not path dependent for homothetic utility. Equation (4) shows that the line integral of income normalized demand is $\ln m^1 - \ln m^0 - V(p^1, m^1) + V(p^0, m^0)$, irrespective the path of integration.

## 4 | THE CONNECTION BETWEEN THE CONSUMER'S INDEX AND CONSUMER'S SURPLUS

For a given pair $(p^0, m^0)$ and $(p^1, m^1)$, Stahl (1983) constructs a utility tracking path of consumer's surplus by tracing the old and new indifference surfaces in price space and joining them by an income jump at an arbitrary price vector. Income also varies along the two indifference surfaces (according to the expenditure function).

For constant elasticity of substitution (CES) utilities, ten Raa (2013, equations 5.12 and 5.13) offers an explicit result by showing that the rectangular path with

$$m = \frac{m^1 - m^0}{\ln m^1 - \ln m^0} \tag{8}$$

makes consumer's surplus (7) track utility (4). He also shows that (8) is an intermediate value of $m^0$ and $m^1$. The main contribution of this paper is that this result holds for any homothetic utility.

**Proposition** *The intermediate income value (8) makes consumer's surplus (7) track utility (4) for any homothetic utility.*

*Proof.* For homothetic utility, equation (5) reduces to $CI = \ln m^1 - \ln m^0 - \int_{p^0}^{p^1} D(p, 1)dp$.

Substitution in equation (7) yields

$$CS = m^1 - m^0 + m(CI - \ln m^1 + \ln m^0). \tag{9}$$

Since $CI$ tracks utility, $CS$ tracks utility if and only if $CS$ has the same sign as $CI$. Substitute ten Raa's CES value (8) for $m$ in $CS$ expression (9). Then expression (9) reduces to

$$CS = m^1 - m^0 + \frac{m^1 - m^0}{\ln m^1 - \ln m^0}(CI - \ln m^1 + \ln m^0) = \frac{m^1 - m^0}{\ln m^1 - \ln m^0}CI = mCI.$$

Because ten Raa (2013, equation 5.13) has shown that $m$ is an intermediate value of $m^0$ and $m^1$, and both are assumed positive, we have $m > 0$. Hence $CS$ with $m$ given by (8) has the same sign as $CI$ and, therefore, tracks utility.

## 5 | CONCLUSION

Consumer's surplus is plagued by the problem of path dependence, even when utility is homothetic. The consumer's index is a dimension-free modification of consumer's surplus. It tracks homothetic utility. Then it singles out the path on which consumer's surplus tracks utility. The use of the consumer's index for non-homothetic utility remains an open question.

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