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A Generalized Expression for the Commodity and the Industry Technology Models in Input–Output Analysis

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ABSTRACT *Technical coefficients are usually constructed from commodity or industry technology models. Although these models are considered as competing, there is an encompassing framework that admits a clear comparison.*

KEY WORDS: Technical coefficients, commodity technology, industry technology, input–output analysis

1. Introduction

Leontief's input–output model features a one-to-one correspondence between industries and products (Steenge, 1990). The matrix of inter-industry flows is square and the resulting input–output table is homogeneous; it can be interpreted as commodity-by-commodity or industry-by-industry table. A first complication comes with the presence of secondary products (by-products, joint products or subsidiary products). During the 1950s and 1960s, rapid industrial diversification caused further problems since input–output tables were being constructed on a single product industrial basis. To cope with

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these problems, the United Nations System of National Accounts (SNA) created two new tables: the so-called use and make tables (recently renamed as use and supply tables in the SNA-93). Although this new framework solved many problems, new problems arose, such as the construction of a technical coefficients matrix on the basis of use and supply matrices (Steenge, 1990).[1] A *use* matrix $\mathbf{U} = (u_{ij})$ comprises commodities i ($=1, \ldots, n$) consumed by sectors j ($=1, \ldots, n$), a *supply* matrix $\mathbf{V} = (u_{ij})$ (formerly the transposed of a make matrix) shows the produce of commodities i in terms of industries j, and an issue is the estimation of the amount of commodity i needed per unit of commodity j.

We formalize the construction of a technical coefficients matrix within a supply-use frame work and derive a general mathematical expression that encompasses the commodity and the industry technology models. We treat the compilation of product by product input–output tables; industry by industry input–output tables may be compiled by assuming fixed product or fixed industry sales structures. This task is on our agenda since industry by industry input–output tables are compiled by several statistical offices, including Denmark, the Netherlands, Norway, Canada, and Finland.

2. Formalization of the Models

Since survey data are based on industry establishments rather than products, we will take as our point of departure the amount of commodity i used by industry j for making either their primary or their secondary outputs (u_{ij}). One must subtract from u_{ij} the consumptions of commodity i used by industry j for its secondary products. Secondary outputs of industry j do not necessarily use the input's structure of their corresponding commodities (as industries could have their own specific input structures). On the other hand, the requirements of commodity i by other industries for making commodity j as secondary output, must be added, if we want to determine the average input requirements of product j, a_{ij}. The total input requirements of commodity i for making commodity j (as a single product) by the economy is thus:

$$u_{ij} - \sum_{\substack{k=1 \\ k \neq j}}^{n} a_{ijk} v_{kj} + \sum_{\substack{k=1 \\ k \neq j}}^{n} a_{ikj} v_{jk}$$

where n is the number of industries as well as of commodities, u_{ij} is the amount of commodity i consumed by industry j, a_{ijk}, the amount of commodity i used by industry j for making each unit of commodity k, a_{ikj}, the amount of commodity i used by other industries k for making each unit of commodity j and v_{kj} (or v_{jk}) the produce of industry j (or k) in terms of commodity k (or j). The first subindex of a_{ijk} denotes commodity inputs (i), the second industries (j), and the third commodity outputs (k).

Bearing in mind that a technical coefficient a_{ij} measures the amount of commodity i used per unit of commodity j, produced by industry j as primary output, we can write:

$$a_{ij} = \frac{u_{ij} - \sum\limits_{\substack{k=1 \\ k \neq j}}^{n} a_{ijk} v_{kj} + \sum\limits_{\substack{k=1 \\ k \neq j}}^{n} a_{ikj} v_{jk}}{\sum\limits_{k=1}^{n} v_{jk}} \tag{1}$$

The problem is that n^2 equations (one for each technical coefficient, a_{ij}) have n^3 unknowns (a_{ijk}, for all $i, j, k = 1,2, \ldots, n$). To solve the equations system (1) assumptions are made, of which the commodity technology and the industry technology assumptions are best known.[2]

The commodity technology hypothesis assumes that all commodities have the same inputs structure irrespective of the industry that produces it. That is, for $j = 1, \ldots, n$

$$a_{ijk} = a_{ik} \tag{2}$$

Under equation (2), equation (1) becomes:

$$a_{ij} = \frac{u_{ij} - \sum_{\substack{k=1\\k\neq j}}^{n} a_{ik}v_{kj} + \sum_{\substack{k=1\\k\neq j}}^{n} a_{ij}v_{jk}}{\sum_{k=1}^{n} v_{jk}} \tag{3}$$

In matrix terms, equation (3) reads:[3]

$$\begin{aligned}\mathbf{A} &= [\mathbf{U} - \mathbf{A}\tilde{\mathbf{V}} + \mathbf{A}\widehat{(\tilde{\mathbf{V}}\mathbf{e})}]\widehat{(\mathbf{V}\mathbf{e})}^{-1} \\ &= [\mathbf{U} - \mathbf{A}\tilde{\mathbf{V}} + \mathbf{A}\widehat{(\tilde{\mathbf{V}}\mathbf{e})}][\widehat{(\mathbf{V}^T\mathbf{e})} + \widehat{(\tilde{\mathbf{V}}\mathbf{e})} - \widehat{(\tilde{\mathbf{V}}^T\mathbf{e})}]^{-1}\end{aligned} \tag{4}$$

Manipulating equation (4) yields:

$$\begin{aligned}\mathbf{U} - \mathbf{A}\tilde{\mathbf{V}} + \mathbf{A}\widehat{(\tilde{\mathbf{V}}\mathbf{e})} &= A\widehat{(\mathbf{V}\mathbf{e})} \\ &= \mathbf{A}[\widehat{(\mathbf{V}^T\mathbf{e})} + \widehat{(\tilde{\mathbf{V}}\mathbf{e})} - \widehat{(\tilde{\mathbf{V}}^T\mathbf{e})}] = \mathbf{A}\widehat{(\mathbf{V}^T\mathbf{e})} + \mathbf{A}\widehat{(\tilde{\mathbf{V}}\mathbf{e})} - \mathbf{A}\widehat{(\tilde{\mathbf{V}}^T\mathbf{e})}\end{aligned}$$

which is the same as:

$$\mathbf{U} - \mathbf{A}\tilde{\mathbf{V}} = \mathbf{A}\widehat{(\mathbf{V}^T\mathbf{e})} - \mathbf{A}\widehat{(\tilde{\mathbf{V}}^T\mathbf{e})} = \mathbf{A}[\widehat{(\mathbf{V}^T\mathbf{e})} - \widehat{(\tilde{\mathbf{V}}^T\mathbf{e})}] = \mathbf{A}(\widehat{\mathbf{V}^T\mathbf{e} - \tilde{\mathbf{V}}^T\mathbf{e}}) = \mathbf{A}\widehat{(\hat{\mathbf{V}}^T\mathbf{e})} = \mathbf{A}\hat{\mathbf{V}}$$

or: $\mathbf{U} = \mathbf{A}\tilde{\mathbf{V}} + \mathbf{A}\hat{\mathbf{V}} = \mathbf{A}(\tilde{\mathbf{V}} + \hat{\mathbf{V}}) = \mathbf{A}\mathbf{V}$. Consequently,

$$\mathbf{A} = \mathbf{U}\mathbf{V}^{-1}. \tag{5}$$

The technical coefficients determined by equation (5) can be negative when the total consumption of input i for the making of secondary outputs of industry j, according to each one of these commodity technologies, exceeds the total use of commodity i by the industry j, either for its primary or secondary products.

The industry technology hypothesis assumes that all industries have the same inputs' structure irrespective of the commodities they produce. This means that for all $k = 1, \ldots, n$:

$$a_{ijk} = a_{ij} \tag{6}$$

and that u_{ij} can be said to be proportional to total industries' outputs, that is,

$$u_{ij} = z_{ij} \sum_{k=1}^{n} v_{kj}$$

or in matrix terms,

$$\mathbf{U} = \mathbf{Z}(\widehat{\mathbf{V}^T\mathbf{e}})$$

Hence, equation (1) becomes:

$$a_{ij} = \frac{u_{ij} - \sum_{\substack{h=1 \\ h \neq j}}^{n} z_{ij} v_{hj} + \sum_{\substack{h=1 \\ h \neq j}}^{n} z_{ih} v_{jh}}{\sum_{h=1}^{n} v_{jh}}$$

which, in matrix terms, is:

$$\mathbf{A} = [\mathbf{U} - \mathbf{Z}(\widehat{\tilde{\mathbf{V}}^T\mathbf{e}}) + \mathbf{Z}\tilde{\mathbf{V}}^T)](\widehat{\mathbf{V}\mathbf{e}})^{-1} \tag{7}$$

or, manipulating equation (7),

$$\mathbf{A} = [\mathbf{U} - \mathbf{U}(\widehat{\mathbf{V}^T\mathbf{e}})^{-1}(\widehat{\tilde{\mathbf{V}}^T\mathbf{e}}) + \mathbf{U}(\widehat{\mathbf{V}^T\mathbf{e}})^{-1}\tilde{\mathbf{V}}^T](\widehat{\mathbf{V}\mathbf{e}})^{-1}$$

and,

$$\begin{aligned}\mathbf{A} &= [\mathbf{U} - \mathbf{U}(\widehat{\mathbf{V}^T\mathbf{e}})^{-1}\{(\widehat{\mathbf{V}^T\mathbf{e}}) - (\widehat{\hat{\mathbf{V}}^T\mathbf{e}})\} + \mathbf{U}(\widehat{\mathbf{V}^T\mathbf{e}})^{-1}\tilde{\mathbf{V}}^T](\widehat{\mathbf{V}\mathbf{e}})^{-1} \\ &= [\mathbf{U} - \mathbf{U} + \mathbf{U}(\widehat{\mathbf{V}^T\mathbf{e}})^{-1}\hat{\mathbf{V}}^T + \mathbf{U}(\widehat{\mathbf{V}^T\mathbf{e}})^{-1}\tilde{\mathbf{V}}^T](\widehat{\mathbf{V}\mathbf{e}})^{-1} \\ &= \mathbf{U}(\widehat{\mathbf{V}^T\mathbf{e}})^{-1}\left[\hat{\mathbf{V}}^T + \tilde{\mathbf{V}}^T\right]\left(\widehat{\mathbf{V}\mathbf{e}}\right)^{-1}\end{aligned}$$

or, since $\tilde{\mathbf{V}}^T = \mathbf{V}^T - \hat{\mathbf{V}}^T$,

$$\mathbf{A} = \mathbf{U}(\widehat{\mathbf{V}^T\mathbf{e}})^{-1}\mathbf{V}^T(\widehat{\mathbf{V}\mathbf{e}})^{-1} \tag{8}$$

Under the industry technology assumption, technical coefficients are non-negative by equation (8).

3. An Empirical Comparison

The general concept of an input–output coefficient a_{ijk}, the amount of commodity i used by industry j for making each unit of commodity k, encompasses the commodity and industry technology models according to restrictions (2) and (6), respectively. The restriction that is most likely in the sense that it best fits the data identifies the more suitable model. Following Mattey and ten Raa (1997), input–output coefficients are considered input regression coefficients of outputs of firm data. Thus, let $l = 1, \ldots, n_j$ be the firms populating industry j. Regress each input i on industry j's outputs:

$$u_{ijl} = \sum_{k=1}^{n} a_{ijk} v_{kjl} + \varepsilon_{ijl} \tag{9}$$

where u_{ijl} and v_{kjl} are the input i and the outputs k of industry j's firm l. The commodity technology hypothesis (2), where the coefficients a_{ik} are given by equation (5), has a p-value, say p_C. For example, if $p_C = 0.2$, then the imposition of the commodity technology assumption pushes the error terms of equation (9) in the tail with 20% mass. Similarly, the industry technology assumption has a p-value, say p_I. For example, if $p_I = 0.3$, the imposition of the industry technology assumption pushes the error terms of equation (9) less, in the tail with 30% mass. In general, a greater p-value indicates a better fit of the technology assumption to the data. Since the input, i, has been fixed in this regression analysis, for some inputs the commodity technology assumption may prove better and for other inputs the industry technology model may be better.

4. Conclusion

The two main methods to construct a technical coefficients matrix within a supply-use framework – the competing commodity and industry technology models – are encompassed by a single formula featuring input–output coefficients a_{ijk}, for the amounts of commodities i used by industries j for making units of commodities k. The two models are represented by alternative restrictions on these industry specific input–output coefficients. This framework is capable of testing which model is more compatible with the data. Following a suggestion by one of the referees, it seems possible to apply the commodity and industry technology approaches to different inputs, thus providing a new form of a mixed technology model.

Notes

[1]Multipliers estimation is another problem mentioned in Steenge (1990) although it is proved in ten Raa and Rueda-Cantuche (2007) and more extensively in Rueda-Cantuche (2004) that employment and output multipliers can be estimated on the basis of use and supply matrices at a micro data level.

[2]See ten Raa and Rueda-Cantuche (2003) for a complete review.

[3]In what follows, **e** will denote a column vector with all entries equal to one, ^{T} will denote transposition and $^{-1}$ inversion of a matrix. Since the latter two operations commute, their composition may be denoted^{-T}. Also, ^ will denote diagonalization, whether by suppression of the off-diagonal elements of a square matrix or by placement of the elements of a vector. ~ will denote a matrix with all the diagonal elements set null.

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