# 10 Bródy's capital

## Introduction

The dynamic input–output (I–O) model reads (Leontief 1970)

$$x = Ax + B\dot{x} + y. \tag{10.1}$$

The left-hand side features the state variable of the economy, the vector of sectoral capacities, as measured by the output levels. The right-hand side lists the material inputs, investment and household demand, respectively. The structure of the economy is given by two matrices of technical coefficients. $A$ is the matrix of input flow coefficients and $B$ is the matrix of input stock coefficients. Input flows, for example electricity, are fully consumed, but input stocks, like housing, carry over. Material inputs are, therefore, proportional to the output levels, but investment is proportional to the new capacity, $\dot{x}$, where the dot denotes the time derivative. Output $x$ and household demand $y$ are functions of time, but the technical coefficients are constant in the absence of structural change. Implicit in the dynamic I–O model is the assumption that productive activity is instantaneous. If you have the commodity vectors $a_{\cdot 1}$ and $b_{\cdot 1}$ (the first columns of technical coefficients matrices $A$ and $B$), then you get instantaneously the commodity vectors $e_1$ and $b_{\cdot 1}$, where $e_1$ is the first unit vector, representing the output flow, and $b_{\cdot 1}$ is the carry-over stock. In ten Raa (1986a) I have dropped this assumption, redefining an input flow coefficient as a time profile on the past; see Chapter 11. Thus, to produce one unit of output 1 at time zero, you need inputs $\alpha_{\cdot 1}(s)$ at times $s, s \leq 0$. The matrix of input flow coefficients, $\alpha$, is a function on the past. It is convenient to work with generalized functions. The dynamic I–O model with instantaneous production is recouped by $\alpha = A\delta$, where $\delta$ is the Dirac function (unit mass in the origin).

A time profile of inputs or outputs that is not concentrated in the origin implies that commodities are tied up in the productive activity. Such commodities constitute capital. The distributed inputs ($\alpha$) represent working capital. In ten Raa (1986b) the analysis is applied to the input stock coefficients, modelling Polish investment lead times by distributing matrix $B$ over the past; see Chapter 12. In a letter, András Bródy wrote to the author:

> Thank you for sending your interesting new paper with computations for the Polish economy. I have to confess I am less enthusiastic about it than I have been with your former papers.
>
> I rather expected you to forge ahead and to drop the notion of a capital matrix, $B$, altogether. You had a distributed input-point output model, already indicating that the inclusion of stocks can be dispensed with. If you generalize to a distributed input-distributed output model, then our traditional approach becomes truly obsolete. Why don't you do it? It is within your reach.

In this chapter, I attempt to take up Bródy's challenge. In other words, let me address the question under which circumstances the dynamic I–O model describes a distributed input-distributed output economy without a preconceived distinction between input flows and stocks.

## The structure of an economy

I maintain the state variable of the economy, $x$, the vector of sectoral capacities. A unit capacity in sector 1 at time zero requires inputs $\alpha_{\cdot 1}(s)$ at times $s, s \leq 0$, and yields outputs $\beta_{\cdot 1}(s)$ at times $s, s \geq 0$. Organizing the sectoral input and output time profiles in a pair of matrix valued (generalized) functions of time, the *structure of the economy* is $(\alpha, \beta)$. Input and output coefficients $\alpha$ and $\beta$ are defined on the past and the future, respectively. All commodities are output of some productive activity. Account for stocks: the capacity of sector 1 at time $t, x_1(t)$, contributes $\beta_{\cdot 1}(s)x_1(t)$ to the economy's commodity stock at time $t + s, s \geq 0$. By change of time variable, the capacity of sector 1 at time $t - s$ contributes $\beta_{\cdot 1}(s)x_1(t - s)$ to the stock at time $t, s \geq 0$. Summing all contributions of the past ($s \geq 0$) by all sectors, we obtain the commodity stock at time $t$, $\int_0^\infty \beta(s)x(t - s)\,\mathrm{d}s$. This is *capital*. Since $\beta$ is confined to the non-negatives, the domain of integration may just as well be extended to the real time. Thus we obtain the convolution product of $\beta$ and $x, \beta * x$, at time $t$. Capital is thus given by the convolution product of the output coefficients, $\beta$, and the sectoral capacities, $x$. The units of capacity are arbitrary, just like the activity levels of von Neumann (1945). Any rescaling is offset by the output coefficients, and the measure of capital is invariant; it is determined by the physical units of the commodities.

Capital at time $t$ can be allocated to future capacity utilization, $x(t - s)$, where $s \leq 0$. In fact, $x(t - s)$ requires commodities $\alpha(s)(t - s)$ at time $t, s \leq 0$. The total allocation of capital at time $t$ to future production amounts $\int_{-\infty}^0 \alpha(s)x(t - s)\,\mathrm{d}s$ or $\alpha * x$ valued at time $t$.

The residual capital constitutes the household stock of commodities, $Y(t)$. Consequently,

$$\int_0^\infty \beta(s)x(t - s)\,\mathrm{d}s = \int_{-\infty}^0 \alpha(s)x(t - s)\,\mathrm{d}s + Y(t) \tag{10.2}$$

or, in short

$$(\beta - \alpha) * x = Y. \tag{10.3}$$

This is the distributed input-distributed output model of the economy. The purpose of this chapter is to determine the structure of the economy for which the dynamic I–O model, (10.1), is submitted by the distributed model with no preconceived flow-stock distribution (10.3).

## The material balance

If the total mass of the net output coefficients, $\int(\beta - \alpha)$, is an $M$-matrix (Minc 1988), then any non-negative stock of household commodities can be sustained by non-negative capacities. However, interesting economics is in terms of flows rather than stocks. The material balance is usually defined in flows, see equation (10.1) for example. The transition to flow is by differentiation of equation (10.2) with respect to time. Differentiating the left-hand side (capital), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \beta(s)x(t-s)\,\mathrm{d}s &= \int_0^\infty \beta(s)\frac{\mathrm{d}}{\mathrm{d}t}x(t-s)\,\mathrm{d}s \\ &= -\int_0^\infty \beta(s)\frac{\mathrm{d}}{\mathrm{d}s}x(t-s)\,\mathrm{d}s \\ &= -\beta(s)x(t-s)|_0^\infty + \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s}\beta(s)x(t-s)\,\mathrm{d}s \\ &= \beta(0)x(t) + \int_0^\infty \dot{\beta}(s)x(t-s)\,\mathrm{d}s \end{aligned} \tag{10.4}$$

where $\dot{\beta}$ is the reduction of output of the capacity units, that is *depreciation*. It takes negative values. Differentiating the first term on the right-hand side of equation (10.2), we get $\int_0^\infty \alpha(s)\dot{x}(t-s)\,\mathrm{d}s$. Lastly, we have $\dot{Y}$, the additions to the household stock of commodities, a commodity flow vector. Thus (10.2) can be rewritten as

$$\beta(0)x(t) = -\int_0^\infty \dot{\beta}(s)x(t-s)\,\mathrm{d}s + \int_0^{-\infty} \alpha(s)\dot{x}(t-s)\,\mathrm{d}s + \dot{Y}(t) \tag{10.5}$$

This is the material balance of the economy. It is a rewrite of the simple distributed input-distributed output model, (10.3). No assumptions have been made. On the left-hand side we have the instantaneous rates of output of current sectoral capacities. On the right-hand side we have, first, replacement investment, second investment in new capacity, and third additions to the household stock.

## The subsumption of dynamic input–output

Replacement investment, $-\int_0^\infty \dot{\beta}(s)x(t-s)\,\mathrm{d}s$, equals the amount of depreciation of capacity $s$ time units old, $-\dot{\beta}(s)$, times the level of capacity at that time, $x(t-s)$, summed over ages $s$. Thus, to determine replacement investment one must know the life pattern of output, $\beta$, and the past distribution of capacity, $x$. In dynamic I–O analysis, one makes the simplifying assumption that one must know only the current stock of capital. The latter is given by the convolution product of $\beta$ and $x$ valued at time $t$ (see second section 'The structure of an economy'). Now it is well known that replacement investment is independent of the age structure of capital ($x$) if and only if the life pattern ($\beta$) is given by the exponential decay function and independent of the sector of installation.

**Assumption 1.** $\beta_{ij}(s) = \beta_{ij}(0)e^{-p_{i}s}$

Then depreciation is, in absolute value,

$$-\dot{\beta} = \rho\beta \tag{10.6}$$

where $\rho$ is the diagonal matrix with elements $\rho_i$. Under Assumption 1, replacement investment is $\rho \int_0^\infty \beta(s)x(t-s)\,\mathrm{d}s$, the direct product of the matrix of depreciation rates and the stock of capital.

As a matter of fact, the dynamic I–O model incorporates replacement investment in the material inputs term, $Ax$. This requires a second step. Replacement investment must not only be proportional to the current stock of capital (irrespective its age structure), but the latter, on its turn, must be proportional to the current level of capacity. In general, the amount of capital allocated to production depends on the entire path of future capacities, through the convolution product with the input time profile, $\alpha$ (see second section 'The structure of an economy'). When is only the current level of capacity relevant? Well, if production takes no time, but is instantaneous:

**Assumption 2.** $\alpha(s) = \alpha(0)\delta(s)$

Here $\delta$ is the Dirac function, the unit mass in the origin. In the 'Introduction' it was noted that this assumption reduces my distributed input model to the traditional one. Thus, what remains crucial now is the distribution of output. Now let us trace the implication of the assumptions on the structure of the economy on the material balance. Substituting Assumption 1's consequence (10.6) and Assumption 2 into balance equation (10.5), we obtain

$$\beta(0)x(t) = \rho \int_0^\infty \beta(s)x(t-s)\,\mathrm{d}s + \alpha(0)\dot{x}(t) + \dot{Y}(t). \tag{10.7}$$

Now replace the capital term by the expression given in equation (10.2) and substitute Assumption 2. Then equation (10.7) becomes

$$\beta(0)x(t) = \rho\alpha(0)x(t) + \alpha(0)\dot{x}(t) + \rho Y(t) + \dot{Y}(t). \tag{10.8}$$

This, then, is the material balance equation under Assumptions 1 and 2. It coincides with the dynamic I–O model, provided secondary products are absent: $\beta(0) = I$, the identity matrix. Note the household demand comprises household stock replacement investment as well as additions to the stock. It is interesting to relate the dynamic I–O coefficients matrices to the structure of the distributed input-distributed output economy:

$$A = \rho\alpha(0), \quad B = \alpha(0) \tag{10.9}$$

Consequently,

$$A = \rho B \tag{10.10}$$

That is, the dynamic I–O model is consistent with a distributed input-distributed output model of the economy if the flow coefficients are *proportional* to the stock coefficients, where the proportions are the rates of depreciation. This relationship between circulating and fixed capital was first noted by Bródy (1974).

I do not wish to subscribe to this relationship as an absolute requirement on technical coefficients matrices. However, an important corollary to the analysis is that if equation (10.10) fails, then the dynamic I–O model, (10.1), is misspecified. The simplifying assumptions made on the structure of the economy will not be valid and one must resort to the general formulation of the material balance: (10.3) (in stocks) or (10.5) (in flows).

## Conclusion

In this chapter, I have attempted to carry out András Bródy's programme to dispense with the distinction between flows and stocks in dynamic I–O analysis by modelling the time profiles of the input and output components of the structure of an economy. The traditional model is retrieved when capital decays exponentially and production is instantaneous. However, the flow and the stock coefficients must be consistent with Bródy's capital equation. If this is not the case, the model better be replaced by the distributed input-distributed output material balance equation derived in this chapter.

## References

Bródy, A. (1974) *Production, Prices and Planning* (North-Holland, Amsterdam).

Leontief, W. (1970) The dynamic inverse. In: A. P. Carter and A. Bródy (eds), *Contributions to Input–Output Analysis* (North-Holland, Amsterdam), pp. 17–46.

Minc, H. (1988) *Nonnegative Matrices* (Wiley, New York).

ten Raa, Th. (1986a) Dynamic input–output analysis with distributed activities, *The Review of Economics and Statistics*, 68(2), 300–10.

ten Raa, Th. (1986b) Applied dynamic input–output with distributed activities, *European Economic Review*, 30(4), 805–31.

von Neumann, J. (1945) A model of general economic equilibrium, *The Review of Economic Studies*, 13, 1–9.