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Consumer surplus and CES demand

By Thijs ten Raa

Tilburg University, Box 90153, 5000 LE Tilburg, the Netherlands; e-mail: tenraa@uvt.nl

Abstract

This article presents the consumer surplus formula for constant elasticity of substitution (CES) demands. The formula is used to compare the monopoly and optimum provisions of product variety. It is shown that a monopolist under-provides variety. This result is contrasted with Lambertini's analysis of the monopolist's optimal R&D portfolio. I also contrast my approach with the indirect utility technique of Anderson, de Palma, and Thisse's discrete choice theory of product differentiation.

JEL classifications: D61, L12

1. Introduction

Alternative measures of consumer's welfare co-exist. There are utility measures and surplus measures. Utility measures are direct (defined on the commodity quantities consumed) or indirect (defined on the commodity prices and the available budget). Surplus measures the difference between the willingness to pay, that is, inverse demand, and price. (This difference is zero at the margin of the last unit purchased of any of the commodities, but positive for the other units purchased.) Marshall (1920) measured surplus using the ordinary demand function, whilst Hicks (1942) did so using the compensated demand functions. The term 'consumer surplus' refers to Marshallian surplus, whilst the Hicksian measures are called equivalent and compensating variations.

Chipman and Moore (1980) provide an elegant consolidation of the alternative welfare measures. Their framework consists of line integrals of vector-valued functions of the prices and the budget. If a vector-valued function yields a path-independent line integral and is non-negative if and only if indirect utility in the terminal point is at least as great as in the initial point, then the function is said to furnish an acceptable integral measure of welfare change. If the function is the gradient of indirect utility (where the derivatives with respect to price can be expressed as the product of demand and the marginal utility of income, by Roy's lemma), we get indirect utility. If the function is demand, we get consumer surplus. If the function is the gradient of an expenditure function, we get a Hicksian variation. Chipman and Moore (1980) investigate if these functions furnish acceptable measures of welfare change. This is trivially true for the first case, indirect utility. It is also true for the Hicksian compensating variation, but not for the Hicksian equivalent variation. The case of consumer surplus has two interesting results. If the budget is constant, then consumer surplus furnishes an acceptable measure of welfare change if and only if utility is homothetic. If one price is constant, then consumer surplus furnishes an acceptable measure of welfare change if and only if utility is quasi-linear, with the linear term representing the commodity of which the price is constant. I consider constant elasticity of substitution (CES) demands with and without a residual commodity, and these two cases will fulfil the respective Chipman and Moore (1980) conditions of homotheticity and quasi-linearity.

Consumer surplus has gone through a roller coaster. It started as a popular welfare measure to evaluate price changes; all you need are ordinary demand functions. However, theorists argued that it is 'inexact' and should be replaced by the Hicksian variations. Willig (1976) restored consumer surplus by showing it is a good approximation. Hausman (1981) showed there is no need for approximation, as the 'exact' concepts can be derived from ordinary demand functions. Takayama (1982) shows that the 'exact' concepts do not measure the welfare impact of a price change correctly, whilst consumer surplus does, at least for homothetic demands. Takayama's result justifies the use of consumer surplus in applied theory.

The main applications of consumer surplus involve linear and log-linear demands, not CES demands. This is surprising, because CES demands are the workhorse of empirical models. CES functions admit the selection of commodities, without reducing utility to zero as for the Cobb-Douglas and Leontief specifications. Moreover, CES functions facilitate tractable analysis, not only of equilibrium prices and quantities under alternative behavioural assumptions, such as Bertrand versus Cournot competition, but also of competitive versus non-competitive imports, via the Armington (1969) specification. However, CES functions are deemed too complicated for surplus measurement. No analytical benefit formula exists according to De Borger (1989, p. 216). Lambertini (2003, p. 563) addresses the monopoly provision of differentiated products by computing producer plus consumer surplus. He stresses that his linear demand parameter is an imperfect indicator of complementarity and suggests that one should use a CES utility function, 'which however would complicate calculations'. Tohamy and Mixon (2004, p. 255) were unable to integrate the CES demand curve and instead used a numerical routine.

This article presents the missing formula and uses it to implement Lambertini's (2003) suggestion to extend his monopoly sub-optimality result to CES demand. This extension result could also be obtained by using the indirect utility technique of Anderson *et al.* (1992). An advantage of consumer surplus over the indirect utility technique is its wider applicability. The moment we depart from the representative consumer implicit in Anderson *et al.* (1992), indirect utility is no longer available, but consumer surplus still is.

Section 2 presents the consumer surplus formula for CES demand. Section 3 shows how a residual commodity, representing the rest of the economy (outside the industry considered) can be accommodated. Section 4 determines the monopoly and optimum provisions of variety. Section 5 contrasts the result with that of Lambertini (2003) and discusses the contribution of the formula vis-à-vis the CES model of Anderson *et al.* (1992).

2. CES consumer surplus

For n commodities the CES utility function is $U(x_1,\ldots,x_n)=(\alpha_1 x_1^\rho+\cdots+\alpha_n x_n^\rho)^{1/\rho}$, where $x_1,\ldots,x_n$ are the commodity quantities, and generates ordinary demands $D_i(p_1,\ldots,p_n;I)$ $=\frac{(\alpha_i/p_i)^\sigma I}{\alpha_1^\sigma p_1^{1-\sigma}+\cdots+\alpha_n^\sigma p_n^{1-\sigma}}$, where $p_1,\ldots,p_n$ are the commodity prices, I is the available budget, and commodity index $i = 1, \ldots, n$. α_i are share parameters. Parameter ρ ranges from minus infinity (Leontief utility) via 0 (Cobb-Douglas utility) to 1 (linear utility). A monotonic transformation is $\sigma = 1/(1-\rho)$, ranging from 0 via 1 to infinity, respectively; σ is the elasticity of substitution. In a monopolistic competition setting σ is also the elasticity of demand. As is customary for consumer surplus calculations, I analyse the effect of a price change of a single commodity, i, from p_i to $p_i{}'$. The variation of consumer surplus is defined by $-\int_{p_i}^{p_i'} D_i(p_1, \ldots, p_{i-1}, p_i'', p_{i+1}, \ldots, p_n; I)dp_i''$. The extension to price vector change is much debated, but straightforward for homothetic demand, as argued by Takayama (1982, p. 40), and trivial for the symmetric demand used in the product variety literature, as shown after the proof of the proposition. The homotheticity ensures that the variation of consumer surplus furnishes an acceptable measure of welfare change in the sense of Chipman and Moore (1980).

Proposition 1 The CES variation of consumer surplus equals $\frac{-I}{1-\sigma}\ln\left\{1+\left[\left(\frac{p_i'}{p_i}\right)^{1-\sigma}-1\right]\frac{p_i x_i}{I}\right\}$.

Let us first check the sign for a price increase, $p_i' > p_i$. If $\sigma < 1$ (>1), the square bracketed expression hence the accolade bracketed expression is positive (negative) and the coefficient is negative (positive), making the product expression negative indeed for both $\sigma < 1$ and $\sigma > 1$.

For $\sigma \to 1$ the logarithm and the denominator both go to 0, but l'Hôpital's rule confirms the Cobb-Douglas variation of consumer surplus, as I will show. By this rule we may take the ratio of the derivatives of the numerator and the denominator (with respect to σ), which yields $\frac{-I}{-1}\left\{1+\left[\left(\frac{p_i'}{p_i}\right)^{1-\sigma}-1\right]\frac{p_i x_i}{I}\right\}^{-1}\frac{p_i x_i}{I}\left(\frac{p_i'}{p_i}\right)^{1-\sigma}(-1)\ln\left(\frac{p_i'}{p_i}\right)$. For $\sigma \to 1$ this expression reduces to $p_i x_i \ln\left(\frac{p_i}{p_i'}\right)$. Indeed, for Cobb-Douglas demand we have $p_i x_i = \alpha_i I$ or $x_i = \frac{\alpha_i I}{p_i}$ with variation of consumer surplus $-\int_{p_i}^{p_i'} D_i(p_i'', I)dp_i'' = -\int_{p_i}^{p_i'} \frac{\alpha_i I}{p_i''}dp_i'' = \alpha_i I \ln\left(\frac{p_i}{p_i'}\right)$, which is perfectly consistent.

Proof of Proposition 1 By definition the variation of consumer surplus is $-\int_{p_i}^{p_i'} D_i\,(p_1, \ldots, p_{i-1}, p_i'', p_{i+1}, \ldots, p_n; I)dp_i'' = -I\int_{p_i}^{p_i'} \frac{(\alpha_i/p_i'')^\sigma}{\alpha_1^\sigma p_1^{1-\sigma}+\cdots+\alpha_i^\sigma p_i''^{1-\sigma}+\cdots+\alpha_n^\sigma p_n^{1-\sigma}}dp_i''$. The primitive function with respect to p_i" is $\frac{1}{1-\sigma}\ln(\alpha_1^\sigma p_1^{1-\sigma}+\cdots+\alpha_i^\sigma p_i''^{1-\sigma}+\cdots+\alpha_n^\sigma p_n^{1-\sigma})$ plus an arbitrary constant, for which I pick $\frac{1}{1-\sigma}\ln(p_i^\sigma x_i/\alpha_i^\sigma)$. The addition of this term is equivalent to replacement of the n terms under the ln in the primitive function by $(\alpha_1/\alpha_i)^\sigma(p_1/p_i)^{-\sigma}p_1x_i, \ldots, (p_i''/p_i)^{-\sigma}p_i''x_i, \ldots, (\alpha_n/\alpha_i)^\sigma(p_n/p_i)^{-\sigma}p_nx_i$. However, since the demand function is generated by budget-constrained utility maximization, the ratio of the marginal utilities equals the price ratio, $(\alpha_1/\alpha_i)(x_1/x_i)^{\rho-1} = p_1/p_i$ or $(\alpha_1/\alpha_i)^\sigma(p_1/p_i)^{-\sigma} = x_1/x_i$, and the first of the latter n terms reduces to p_1x_1 and similar for the other terms; only the ith term stays $(p_i''/p_i)^{-\sigma}p_i''x_i = p_i''^{1-\sigma}p_i{}^\sigma x_i$, where $p_i'' = p_i'$ or p_i (the upper and lower bounds of the integral). Summing these terms the expression under the ln function reads $I + (p_i''^{1-\sigma}p_i{}^\sigma - p_i)x_i$ and, therefore, the variation of consumer surplus becomes $\frac{-I}{1-\sigma}\ln\frac{I+(p_i'^{1-\sigma}p_i{}^\sigma-p_i)x_i}{I} = \frac{-I}{1-\sigma}\ln\left\{1+\left[\left(\frac{p_i'}{p_i}\right)^{1-\sigma}-1\right]\frac{p_ix_i}{I}\right\}$. This completes the proof.

Corollary 1 In the symmetric case, with equal coefficients, the CES demand functions reduce to $D_i(p_1,\ldots,p_n;I)=\frac{I/p_i^\sigma}{p_1^{1-\sigma}+\cdots+p_n^{1-\sigma}}$. For a symmetric price vector change, from $(p,\ldots,p)$ to $(p',\ldots,p')$, the variation in consumer surplus is $-I\ln\frac{p'}{p}$, as shown in Appendix 1.

Remark The budget share in the formula is $\frac{p_i x_i}{I}=\frac{(\alpha_i/p_i)^\sigma}{\alpha_1^\sigma p_1^{1-\sigma}+\cdots+\alpha_n^\sigma p_n^{1-\sigma}}$. The price elasticity is unitary, which explains that the variation of consumer surplus is logarithmic and not summable. Hence consumer surplus at p_i (with reference price p_i' infinity) is not defined. This problem does not occur when there is residual income.

3. CES consumer surplus with residual demand

For n commodities with residual demand the CES utility function is $U(x_0,x_1,\ldots,x_n)=x_0+(\alpha_1 x_1^\rho+\cdots+\alpha_n x_n^\rho)^{m/\rho}$. The quasi-linearity ensures that consumer surplus will furnish an acceptable measure of welfare change in the sense of Chipman and Moore (1980).

If $m=1$, utility is homothetic and demand will be concentrated in either the differentiated commodities or the residual commodity, bringing us back to the previous section. From this section onwards I assume that $0<m<1$. Then demands for both commodity types are positive. The available budget, say $\underline{I}$, is now divided between expenditure I on the differentiated commodities and residual demand x_0, where numeraire residual demand has been scaled such that its price is 1. The demand for the differentiated commodities are $D_i(p_1,\ldots,p_n;I)=\frac{(\alpha_i/p_i)^\sigma I}{\alpha_1^\sigma p_1^{1-\sigma}+\cdots+\alpha_n^\sigma p_n^{1-\sigma}}$. The expenditure on the differentiated commodities or, equivalently, the residual demand, is determined by the condition that the marginal utility of the bundle of differentiated commodities with respect to expenditure equals 1. The differentiated commodities contribute utility $(\alpha_1^\sigma p_1^{1-\sigma}+\cdots+\alpha_n^\sigma p_n^{1-\sigma})^{\frac{m}{\sigma-1}}I^m$. Differentiating with respect to I the optimality condition reads $m(\alpha_1^\sigma p_1^{1-\sigma}+\cdots+\alpha_n^\sigma p_n^{1-\sigma})^{\frac{m}{\sigma-1}}$ $I^{m-1}=1$. Solving, the expenditure on the differentiated commodities is $I(p_1,\ldots,p_n)=m^{1/(1-m)}(\alpha_1^\sigma p_1^{1-\sigma}+\cdots+\alpha_n^\sigma p_n^{1-\sigma})^{\frac{1}{(1/m-1)(\sigma-1)}}$. Substituting back, demand becomes $D_i=m^{1/(1-m)}(\alpha_i/p_i)^\sigma(\alpha_1^\sigma p_1^{1-\sigma}+\cdots+\alpha_n^\sigma p_n^{1-\sigma})^{\frac{1/(1-m)-\sigma}{\sigma-1}}$.

Proposition 2 With residual demand CES consumer surplus equals $(1/m-1)m^{1/(1-m)}(\alpha_1^\sigma p_1^{1-\sigma}+\cdots+\alpha_n^\sigma p_n^{1-\sigma})^{\frac{1}{(1/m-1)(\sigma-1)}}$.

Proof of Proposition 2 Consumer surplus is $m^{1/(1-m)}\int_{p_i}^{\infty}(\alpha_1^\sigma p_1^{1-\sigma}+\cdots+\alpha_i^\sigma p_i'^{1-\sigma}+\cdots+\alpha_n^\sigma p_n^{1-\sigma})^{\frac{1/(1-m)-\sigma}{\sigma-1}}(\alpha_i/p_i')^\sigma dp_i'$. The primitive function with respect to p_i' is $\frac{1}{1-\sigma}\frac{1/m-1}{1/\rho-1}(\alpha_1^\sigma p_1^{1-\sigma}+\cdots+\alpha_i^\sigma p_i'^{1-\sigma}+\cdots+\alpha_n^\sigma p_n^{1-\sigma})^{\frac{1}{(1/m-1)(\sigma-1)}}=(1-1/m)(\alpha_1^\sigma p_1^{1-\sigma}+\cdots+\alpha_i^\sigma p_i'^{1-\sigma}+\cdots+\alpha_n^\sigma p_n^{1-\sigma})^{\frac{1}{(1/m-1)(\sigma-1)}}$. This behaves like $p_i'^{(1-\sigma)\frac{1/\rho-1}{1/m-1}}=p_i'^{\frac{1}{1-1/m}}$, which tends to 0 since $m<1$. Hence consumer surplus is $m^{1/(1-m)}(1/m-1)(\alpha_1^\sigma p_1^{1-\sigma}+\cdots+\alpha_i^\sigma p_i^{1-\sigma}+\cdots+\alpha_n^\sigma p_n^{1-\sigma})^{\frac{1}{(1/m-1)(\sigma-1)}}$.

Corollary 2 In the symmetric case, $\alpha_i=\alpha$ and $p_i=p$, demand is $D_i=m^{1/(1-m)}(\alpha/p)^\sigma(n\alpha^\sigma p^{1-\sigma})^{\frac{1/(1-m)-\sigma}{\sigma-1}}$. The price elasticity is $-1/(1-m)$, which is independent of the elasticity of substitution. The elasticity of substitution does affect the level of demand (through the power of α, namely, $\sigma\left(1+\frac{1/(1-m)-\sigma}{\sigma-1}\right)=\frac{\sigma}{\sigma-1}\left(-1+\frac{1}{1-m}\right)=\frac{1/\rho}{1/m-1}$).

Complementarity (as measured by a lower value of ρ or, equivalently, σ) boosts demand and hence consumer surplus (for $\alpha > 1$, as will be the case). Consumer surplus reduces to $(1/m-1)m^{1/(1-m)}n^{\frac{1}{(1/m-1)(\sigma-1)}}\alpha^{\frac{\sigma}{(1/m-1)(\sigma-1)}}/p^{\frac{1}{1/m-1}}$.

4. Monopoly and optimum provision of variety

Dixit and Stiglitz (1977, p. 301) 'have a rather surprising case where the monopolistic competition equilibrium is identical with the optimum constrained by the lack of lump sum subsidies'. In their case utility is a function of the numeraire and the symmetric CES aggregate $(x_1^\rho + \cdots + x_n^\rho)^{1/\rho}$. The number of varieties, n, is determined by market forces or a nonnegative profit constrained welfare optimization. Consumers have taste for variety. The utility gain derived from spreading a unit of production between n differentiated products is $[(1/n)^\rho + \cdots + (1/n)^\rho]^{1/\rho}/1 = [n(1/n)^\rho]^{1/\rho} = n^{(1-\rho)/\rho}$. Benassy (1996) defines the elasticity of the taste variety in the obvious way and thus shows it equals to $1/\rho - 1$ in the Dixit-Stiglitz framework. The elasticity of demand is $\sigma = 1/(1-\rho)$ and monopolistic profit maximization yields that the mark-up relative to price is the inverse elasticity $\frac{p-c}{p} = \sigma^{-1}$ or that the mark-up relative to cost c is $\frac{p-c}{c} = \frac{p}{c} - 1 = (1-\sigma^{-1})^{-1} - 1 = [1-(1-\rho)]^{-1} - 1 = 1/\rho - 1$. Benassy (1996) shows that this coincidence between taste for variety and mark-up explains the Dixit-Stiglitz result and introduces a multiplicative power function of n in the CES aggregate to control for the variety elasticity independent of the demand elasticity or mark-up.

In the symmetric Dixit-Stiglitz model consumers maximize $U(\underline{I} - npx, (x^\rho + \cdots + x^\rho)^{1/\rho}) = U(\underline{I} - npx, n^{1/\rho}x)$. Denoting partial derivatives by subscripts 0 for numeraire and 1 for the differentiated commodities, the first-order condition is that the marginal rate of substitution is $U_1/U_0 = n^{1-1/\rho}p$. If U is Cobb-Douglas with coefficients α for the differentiated commodity aggregate and $1-\alpha$ for the numeraire, as in Anderson *et al.* (1992), we have $npx = \alpha\underline{I}$. The elasticity of demand is 1, and the monopoly price becomes infinite. If U is linear (as in partial equilibrium analysis), the marginal rate of substitution is constant and can be set equal to 1 by choice of differentiated commodities unit, so that utility is $\underline{I} - npx + n^{1/\rho}x$. The consumer maximizes $(n^{1/\rho} - np)x$; if p is greater than $n^{1/\rho-1}$ demand is 0 and otherwise the budget will be exhausted: $npx = \underline{I}$. Since revenue is constant the monopolist minimises cost by offering no variety, $n = 1$. To avoid this bang-bang behaviour featuring uninteresting monopoly solutions, I raise the CES aggregate to a power m, a constant between 0 and 1, so that for any given number of varieties there is diminishing marginal utility. I also include a power of n to control for the taste of variety.

The utility function I use is $U(x_0, x_1, \ldots, x_n) = x_0 + n^{\nu+m-m/\rho}(x_1^\rho + \cdots + x_n^\rho)^{m/\rho}$. Here the multiplicative coefficient has been chosen following Benassy (1996): if the prices are equal and the expenditure is split accordingly, then the contribution to utility is $n^{\nu+m-m/\rho}\left[\left(\frac{I}{np}\right)^\rho + \cdots + \left(\frac{I}{np}\right)^\rho\right]^{m/\rho} = n^{\nu+m-m/\rho}\left[n\left(\frac{I}{np}\right)^\rho\right]^{m/\rho} = n^\nu (I/p)^m$, which has variety elasticity ν.

4.1 Monopoly variety

The budget constraint of the consumer is binding and can be used to eliminate residual income x_0. The constant income term may be deleted and utility reduces to

$n^{\nu+m-m/\rho}(nx^\rho)^{m/\rho} - npx = n^{\nu+m}x^m - npx$. The first-order demand condition (with respect to x) can be rewritten as $npx = n^{\nu+m}mx^m$. This is revenue. The implicit assumption is that income is greater than this expression, ruling out the corner solution where all income is spent on the differentiated commodities offered by the monopolist. (Because we will solve for quantities and variety, the lower bound of income is a function of the parameters.) Dividing revenue by np and rewriting we see that demand is $x = (n^{\nu+m-1}m/p)^{1/(1-m)}$. For each variety let the set-up cost be F and the marginal cost be c. Then profit is $n^{\nu+m}mx^m - cnx - nF$. The first-order condition with respect to x yields $x = (n^{\nu+m-1}m^2/c)^{1/(1-m)}$. The first-order condition with respect to n is $(\nu+m)n^{\nu+m-1}mx^m - cx - F = 0$, ignoring the integer problem (following Dixit and Stiglitz, 1977; Anderson *et al.*, 1992; Benassy, 1996). I assume that there is taste for variety, but with diminishing returns: the elasticity fulfils the following maintained.

Assumption The variety elasticity, ν, is less than 1 – m, where m is the utility elasticity of the differentiated products.

This ensures the first-order conditions solve, barring bang-bang bevaviour. In fact, the monopoly variety is $n = \left(\frac{\nu m^{(1+m)/(1-m)}}{Fc^{m/(1-m)}}\right)^{\frac{1-m}{1-m-\nu}}$. The derivation is in Appendix 2.

4.2 Second-best variety

As I confirm later, in the first-best solution price equals marginal cost, hence profit is negative. In the second-best solution producer surplus must be at least 0 (and this constraint is binding). From Section 4.1 we see that the zero-profit condition reads $n^{\nu+m}mx^m - cnx - nF = 0$. It implies a marginal rate of substitution (between variety n and quantity x) of $\frac{(\nu+m)n^{\nu+m-1}mx^m - cx - F}{n^{\nu+m}m^2x^{m-1} - cn} = \frac{(\nu+m-1)n^{\nu+m-1}mx^m}{n^{\nu+m}m^2x^{m-1} - cn}$, where we substituted the zero-profit condition. By Corollary 2, with $\alpha^{m/\rho} = n^{\nu+m-m/\rho}$, or $\alpha^\sigma = n^{(\nu+m-m/\rho)\rho\sigma/m} = n^{[(\nu+m)(\sigma-1)-m\sigma]/m} = n^{\nu(\sigma-1)/m-1}$, consumer surplus reduces to $(1/m-1)m^{1/(1-m)}n^{\frac{\nu}{1-m}}/p^{\frac{1}{1/m-1}}$. Substituting inverse demand, which is readily available from revenue expression $npx = n^{\nu+m}mx^m$, and simplifying, consumer surplus can be rewritten as $(1/m-1)m^{1/(1-m)}n^{\frac{\nu}{1-m}}/(n^{\nu+m-1}mx^{m-1})^{\frac{1}{1/m-1}} = (1-m)n^{\nu+m}x^m$. It implies a marginal rate of substitution (between n and x) of $\frac{(\nu+m)/n}{m/x} = (1+\nu/m)x/n$.

The first-order condition of the maximization of consumer surplus subject to the producer surplus constraint is that the marginal rates of substitution of producer and consumer surplus are equal. Solving, $x^{1-m} = \frac{n^{\nu+m-1}m}{(1+\nu/m)c}$. Comparison with the demand relationship, $npx = n^{\nu+m}mx^m$, yields $p = (1+\nu/m)c$. By the zero-profit condition the mark-up offsets the fixed cost: $(\nu/m)cx = F$. Hence $x = mF/\nu c$, which is independent of n. Combining the last two expressions for x, $(mF/\nu c)^{1-m} = \frac{n^{\nu+m-1}m}{(1+\nu/m)c}$. Hence the second-best variety is $n = \left(\frac{m^m\nu^{1-m}}{F^{1-m}c^m(1+\nu/m)}\right)^{\frac{1}{1-m-\nu}}$.

4.3 First-best variety

The derivative of consumer surplus with respect to p_i is minus the demand for that commodity, $-D_i$. The derivative of producer surplus $(p_1-c)D_1 + \cdots + (p_n-c)D_n - nF$ with respect to p_i is $D_i + (p_1-c)\partial D_1/\partial p_i + \cdots + (p_n-c)D_n/\partial p_i$. Setting the sum of the derivatives equal to 0, the first-order condition of total surplus maximization with respect to price reads $(p_1-c)\partial D_1/\partial p_i + \cdots + (p_n-c)D_n/\partial p_i = 0$, $i = 1, \ldots, n$. By symmetry and (differentiated commodities) demand homogeneity (of degree $1/(m-1) < 0$), we conclude the familiar first best condition $p_i = c$. By Section 4.2 consumer surplus is equal to $(1/m-1)m^{1/(1-m)}n^{\frac{\nu}{1-m}}/c^{\frac{1}{1/m-1}}$ and producer surplus is $-nF$. The first-order condition for total surplus maximization is $(1/m-1)m^{1/(1-m)}\frac{\nu}{1-m}n^{\frac{\nu}{1-m}-1}c^{\frac{-1}{1/m-1}} = F$. Solving, $n = \left(\frac{\nu^{1-m}m^m}{F^{1-m}c^m}\right)^{\frac{1}{1-m-\nu}}$.

The explicit variety solutions for the monopoly, second-best, and first-best cases admit direct comparisons.

Proposition 3 If the variety elasticity, ν, is less than $1-m$, where m is the utility elasticity of the differentiated products, then a monopolist under-provides variety relative to the second-best variety, which in turn is smaller than the first-best variety.

Proof of Proposition 3 From the results of Sections 4.1 and 4.2 we see that the ratio of the monopoly variety to the second best variety is $\left(\frac{\nu m^{(1+m)/(1-m)}}{Fc^{m/(1-m)}}\right)^{\frac{1-m}{1-m-\nu}}/\left(\frac{m^m\nu^{1-m}}{F^{1-m}c^m(1+\nu/m)}\right)^{\frac{1}{1-m-\nu}} = (m+\nu)^{\frac{1}{1-m-\nu}} < 1$, because $m+\nu<1$ (and therefore the exponent is positive), by assumption. From the results of Sections 4.2 and 4.3 we see that the ratio of the second-best to the first-best variety is $\left(\frac{m^m\nu^{1-m}}{F^{1-m}c^m(1+\nu/m)}\right)^{\frac{1}{1-m-\nu}}/\left(\frac{\nu^{1-m}m^m}{F^{1-m}c^m}\right)^{\frac{1}{1-m-\nu}} = 1/(1+\nu/m)^{\frac{1}{1-m-\nu}} < 1$ (because the exponent is positive). This completes the proof.

5. Discussion

Proposition 3 is closely related to Lambertini's (2003) analysis of the monopoly provision of variety when demand is linear. When the commodities are substitutes we are in perfect agreement: a monopolist under-provides variety. When the commodities are complements I again obtain under-provision, but Lambertini (2003) finds that the provision is just right. One may question the robustness of this result, as Benassy (1996) questioned the robustness of the Dixit and Stiglitz (1977) result that the monopolistic competition equilibrium is identical with the second-best optimum. To be fair, Lambertini (2003) himself acknowledged that his linear demand is an imperfect model of complementarity. In fact, LaFrance (1985) has shown that the underlying utility function must be quasi-linear, with a quadratic nonlinear term and with the linear term representing the residual numeraire commodity. Lambertini's (2003) utility function captures the quadratic term, but misses the residual income term.

Anderson *et al.* (1992) use a CES utility function to compare optimum and monopolistic equilibrium product diversity of substitutes. They do capture the numeraire commodity, albeit in Cobb-Douglas, not quasi-linear fashion. Consumer surplus would not furnish an acceptable measure of welfare change in the sense of Chipman and Moore (1980). Instead, Anderson *et al.* (1992) derive and use the indirect utility function. Their (Cobb-Douglas) specification is not applicable to the monopoly problem addressed in this article—the monopoly price level would become infinite—but in principle I could have used their technique. This procedure would amount to an alternative proof of Proposition 3. The reason of this consistency is that the Chipman and Moore (1980) result renders the indirect utility and consumer surplus approaches equivalent for our quasi-linear CES function. However, an advantage of consumer surplus over the indirect utility technique is its wider applicability. The moment we depart from the representative consumer implicit in Anderson *et al.* (1992), indirect utility is no longer available, but consumer surplus still is. If the number of household types becomes as large as the number of varieties, all we can say is that demand fulfills Walras's law (Debreu, 1974). For example, in applied general equilibrium analysis the different household types are modelled by different CES demands. Consumer surplus can be evaluated, but not so indirect utility.

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References

Anderson, S.P., de Palma, A., and Thisse, J.-F. (1992) *Discrete Choice Theory of Product Differentiation*, MIT Press, Cambridge, MA.

Armington, P.S. (1969) A theory of demand for products distinguished by place of production, *International Monetary Fund Staff Papers*, **16**, 159–78.

Benassy, J.-P. (1996) Taste for variety and optimum production patterns in monopolistic competition, *Economics Letters*, **52**, 41–7.

Chipman, J.S. and Moore, J.C. (1980) Compensating variation, consumer's surplus, and welfare, *American Economic Review*, **70**, 933–49.

De Borger, B. (1989) Estimating the welfare implications of in-kind government programs, *Journal of Public Economics*, **38**, 215–26.

Debreu, G. (1974) Excess demand functions, *Journal of Mathematical Economics*, **1**, 15–21.

Dixit, A.K. and Stiglitz, J.E. (1977) Monopolistic competition and optimum product diversity, *American Economic Review*, **67**, 297–308.

Hausman, J.A. (1981) Exact consumer's surplus and deadweight loss, *American Economic Review*, **71**, 662–76.

Hicks, J.R. (1942) Consumers' surplus and index-numbers, *Review of Economic Studies*, **9**, 126–37.

LaFrance, J.T. (1985) Linear demand functions in theory and practice, *Journal of Economic Theory*, **37**, 147–66.

Lambertini, L. (2003) The monopolist's optimal R&D portfolio, *Oxford Economic Papers*, **55**, 561–78.

Marshall, A. (1920) *Principles of Economics*, Macmillan, London.

Takayama, A. (1982) On consumer's surplus, *Economics Letters*, **10**, 35–42.

Tohamy, S.M. and Mixon, J.W. Jr. (2004) Illustrating consumer theory with the CES utility function, *Journal of Economic Education*, **35**, 251–8.

Willig, R.D. (1976) Consumer's surplus without apology, *American Economic Review*, **66**, 589–97.

Appendix 1: Proof of Corollary 1

The variation of consumer surplus can be evaluated by changing one price at a time. For a symmetric price vector change, from $(p, \ldots, p)$ to $(p', \ldots, p')$, the contribution to the variation of consumer surplus due to the first price component change is $-\int_p^{p'} D_1(p'', p, \ldots, p; I)dp'' = \frac{-I}{1-\sigma}\ln\left\{1 + \left[\left(\frac{p'}{p}\right)^{1-\sigma} - 1\right]\frac{pD_1(p,p,\ldots,p;I)}{I}\right\}$, with the argument of the ln equal to $1 + \left[\left(\frac{p'}{p}\right)^{1-\sigma} - 1\right]\frac{p^{1-\sigma}}{p^{1-\sigma}+\cdots+p^{1-\sigma}} = 1 + \frac{p'^{1-\sigma}-p^{1-\sigma}}{p^{1-\sigma}+\cdots+p^{1-\sigma}} = \frac{p'^{1-\sigma}+p^{1-\sigma}\cdots+p^{1-\sigma}}{p^{1-\sigma}+\cdots+p^{1-\sigma}}$.

The contribution due to the second price change is $-\int_p^{p'} D_2(p', p'', p, \ldots, p; I)dp'' = \frac{-I}{1-\sigma}\ln\left\{1 + \left[\left(\frac{p'}{p}\right)^{1-\sigma} - 1\right]\frac{pD_2(p', p, \ldots, p; I)}{I}\right\}$, with the argument of the ln equal to $1 + \left[\left(\frac{p'}{p}\right)^{1-\sigma} - 1\right]\frac{p^{1-\sigma}}{p'^{1-\sigma}+p^{1-\sigma}+\cdots+p^{1-\sigma}} = 1 + \frac{p'^{1-\sigma}-p^{1-\sigma}}{p'^{1-\sigma}+p^{1-\sigma}+\cdots+p^{1-\sigma}} = \frac{p'^{1-\sigma}+p'^{1-\sigma}+p^{1-\sigma}\cdots+p^{1-\sigma}}{p'^{1-\sigma}+p^{1-\sigma}+\cdots+p^{1-\sigma}}$.

And so on. The contribution due to the nth price change is $-\int_p^{p'} D_n(p', \ldots, p', p''; I)dp'' = \frac{-I}{1-\sigma}\ln\left\{1 + \left[\left(\frac{p'}{p}\right)^{1-\sigma} - 1\right]\frac{pD_n(p', \ldots, p', p; I)}{I}\right\}$, with the argument of the ln equal to $1 + \left[\left(\frac{p'}{p}\right)^{1-\sigma} - 1\right]\frac{p^{1-\sigma}}{p'^{1-\sigma}+\cdots+p'^{1-\sigma}+p^{1-\sigma}} = 1 + \frac{p'^{1-\sigma}-p^{1-\sigma}}{p'^{1-\sigma}+\cdots+p'^{1-\sigma}+p^{1-\sigma}} = \frac{p'^{1-\sigma}+\cdots+p'^{1-\sigma}+p'^{1-\sigma}}{p'^{1-\sigma}+\cdots+p'^{1-\sigma}+p^{1-\sigma}}$.Summing the n contributions to the variation of consumer surplus we obtain $\frac{-I}{1-\sigma}$ times the sum of the n natural logarithms. That sum is the ln of the product of the n arguments. Because the denominator of an argument cancels against the numerator of the preceding argument, only the first denominator and last numerator remain. Hence the variation in consumer surplus is $\frac{-I}{1-\sigma}\ln\frac{p'^{1-\sigma}+\cdots+p'^{1-\sigma}}{p^{1-\sigma}+\cdots+p^{1-\sigma}} = -I\ln\frac{p'}{p}$.

Appendix 2: Derivation of the monopoly number of varieties

The first-order conditions of the interior solution are $x = (n^{\nu+m-1}m^2/c)^{1/(1-m)}$ and $(\nu+m)n^{\nu+m-1}mx^m - cx - F = 0$. Substituting the former into the latter, we obtain $(\nu+m)n^{\nu+m-1}m(n^{\nu+m-1}m^2/c)^{m/(1-m)} - c(n^{\nu+m-1}m^2/c)^{1/(1-m)} - F = 0$. Collecting powers, $c(1/c)^{1/(1-m)} = (1/c)^{1/(1-m)-1} = (1/c)^{m/(1-m)}$ and $n^{\nu+m-1}(n^{\nu+m-1})^{m/(1-m)} = (n^{\nu+m-1})^{1/(1-m)} = n^{(\nu+m-1)/(1-m)}$, we obtain $[(\nu+m)m(m^2)^{m/(1-m)} - (m^2)^{1/(1-m)}](1/c)^{m/(1-m)}n^{(\nu+m-1)/(1-m)} - F = 0$. Since the bracketed expression is simply $\nu m^{(1+m)/(1-m)} + m^{2/(1-m)} - m^{2/(1-m)} = \nu m^{(1+m)/(1-m)}$, we conclude $\nu m^{(1+m)/(1-m)}(1/c)^{m/(1-m)}n^{(\nu+m-1)/(1-m)} - F = 0$ or $n = \left(\frac{Fc^{m/(1-m)}}{\nu m^{(1+m)/(1-m)}}\right)^{\frac{1-m}{\nu-(1-m)}} = \left(\frac{\nu^{1-m}m^{1+m}}{F^{1-m}c^m}\right)^{\frac{1}{1-m-\nu}}$.