CHAPTER 12

AN ALTERNATIVE TO DEBREU'S DATED AND LOCATED COMMODITIES (OR THE ECONOMY AS AN ONION)

Thijs ten Raa
Erasmus University, Rotterdam, The Netherlands

1. DATED AND LOCATED COMMODITIES

The usual framework of general equilibrium analysis, since the inception of Debreu (1959), is an ℓ–dimensional Euclidean space with one dimension for each commodity. Thus there are ℓ commodities and with each one a single real number is associated: its quantity.

Now let the real world contain n physically different goods and services. Debreu's example is wheat. What is the quantity of wheat? To avoid infinity, we must consider a quantity of wheat per unit of time. But then we have a variety of wheat quantities, one for each date. To meet the condition that a single real number is associated with a commodity, we must define wheat at a certain date as a distinct commodity. That is, commodities are dated. Let the number of dates be T. Then there are nT commodities, so far.

Another complication is that wheat in one location, say Kansas, cannot be the same commodity as wheat in another, say New York. For if it were, transportation would be nonsense, on the assumption that one can always produce a commodity out of itself by doing nothing. Therefore, commodities are also located. Let the number of locations be S. Then the total number of commodities equals the number of goods multiplied with the numbers of dates and locations: $\ell = nTS$.

Note that since the commodity space is ℓ–dimensional Euclidean, ℓ is finite, and, therefore, T and S too. Time and space must be discrete and have finite horizons. We end up with one very long list of commodities, $1, 2, 3, \ldots, \ell$. These commodities are independent actors in the general equilibrium performance. Whether they are differentiated by physical, temporal or spatial pecularities is inessential.

2. DISCUSSION

The economy is much like an onion. It has a kaleidoscopic structure as that of spheres, rings and segments. Adopting alternative points of view one can form a mosaic of beautiful patterns. The general equilibrium analysist, however, peels the onion apart to separate layers of different vintages which, on their turn, are cut into pieces of different locations. To make sure that no parts remain connected, the remainders are thoroughly chopped up. The smahed substance is served and swallowed down. Since mashed onions are not exactly a delicacy, a strong wine must finish off digestion. A Chateau Brouwer of 1910 serves the purpose quite well.

The draw-back of dating and locating goods is, paradoxally, the destruction of time and space. Time and space are reduced to an index set of date and location labels for commodities. The index set has no structure. The order of the dates and locations is arbitrary. Once the onion is chopped, its structure is lost. Temporal and spatial subtleties such as accumulation or diffusion are treated in the same way as differences between forks and knives.

Our comprehension is lessened too. Initially we recognised a whole nest of spheres in the onion, but now there is just a mess of tiny bits. The original economy consisted of n goods plus time and space. But the resulting model comprises numerous commodities, namely $\ell = nTS$. And, if equilibrium between supply and demand is analysed for all commodities, the number of equations is equally sizeable.

These considerations motivate our search for an alternative framework for economic analysis. The new space should facilitate whole goods, including their temporal and spatial extensions. It must be richer than Euclidean space to avoid dating and locating. We should be able to associate a whole temporal and/or spatial distribution of quantities with a good, rather than just a single real number. Therefore, it is natural to propose as an alternative framework the space of n-vector valued distributions over time and/or space. The introduction of the new framework is simplified by confining ourselves, for the time being, to time, and considering a Leontief technology.

3. TIME

Consider a dynamic economy with n goods. Let $y(t)$ be the n-vector of quantities supplied at time t. Then $\dot{y}(t)$ is the vector of accumulation rates at time t, dot denoting differentiation with respect to time. This requires investment of $B\dot{y}(t)$ at time t, assuming fixed capital coefficients which are arranged in the nxn-matrix B. Let $x(t)$ be the n-vector of quantities demanded at time t, other than for investment. Then the condition of equilibrium is

$$y(t) = B\dot{y}(t) + x(t).$$

Since this must hold for all t, we obtain

$$y = B\dot{y} + x.$$

This is an n-vector valued equation of equilibrium of simple form. The dating device consists of restricting t to the index set $\{1, ..., T\}$ and to define the T y(t)'s as distinct commodity vectors. $\dot{y}(t)$ is replaced by $y(t+1) - y(t)$. Then the equation becomes

$$\begin{pmatrix} y(1) \\ \cdot \\ \cdot \\ \cdot \\ y(T) \end{pmatrix} = A \begin{pmatrix} y(1) \\ \cdot \\ \cdot \\ \cdot \\ y(T) \end{pmatrix} + \begin{pmatrix} x(1) \\ \cdot \\ \cdot \\ \cdot \\ x(T) \end{pmatrix}$$

where the nTxnT-matrix

$$A = \begin{pmatrix} -B & B & & 0 \\ & \ddots & \ddots & \\ & & \ddots & B \\ 0 & & & -B \end{pmatrix}$$

This system can be solved for (y(1) . . . y(T)) given (x(1) . . . x(T)). Note that the analysis is essentially the same as that of a static economy with nT commodities arranged in one long vector which is subjected to an input–output matrix. Formally, the commodity space is R^{nT} and technology is in R^{nTxnT}, applying through the ordinary matrix product.

Alternatively, we consider y itself as an element of a commodity space. This space, then, is, heuristically, the space of functions from R (time) to R^n (quantities of goods). We preserve n, the number of goods. There is, however, a problem in placing the equation of equilibrium. The first term is clear: it is an element of the commodity space. But the second term, $B\dot{y}$, is opaque. Is B a technology and $\dot{y}$ a commodity? Our approach to clarification of the meaning of the terms is to generalise the notion of a function in the sense of Schwartz (1957). Formally, we take $D'(R,R^n)$ as commodity space, i.e. the space of n-vector generalised functions, also called distributions, over time. The generalisation of technology is similar: we position it in $D'(R,R^{nxn})$, the space of nxn-matrix distributions over time.

Now let us check the equation of equilibrium. The first term remains, albeit reinterpreted as a distribution. The second term reflects an application of technology on the commodity vector distribution. The application is immediate, no lags are involved. That is, technology is concentrated in time, like $B\delta$, where B is an ordinary nxn-matrix and δ is the unit point mass on the time axis, i.e. Dirac distribution. Introducing the convolution product, $*$, extended in component by component fashion to matrix distributions, our second term becomes $B\delta*\dot{y}$. (δ is the unit element of the convolution product; heuristically $(B\delta*\dot{y})(t) = \int B\delta(s)\dot{y}(t-s)\,ds = B\delta(0)\,\dot{y}(t-0) = B\dot{y}(t)$. For sound mathematical detail see Schwartz (1957).) The differentiation dot may be transferred to the first factor. (This elementary fact is established by partial integration). The whole term, $B\dot{\delta}*y$, now consists of a technical nxn-matrix distribution applying through the convolution product on the original commodity space element. In sum, we have commodities x and y in $D'(R,R^n)$, a technology $B\dot{\delta}$ in $D'(R,R^{nxn})$, and an equation for their equilibrium interrelationship:

$$y = B\dot{\delta}*\dot{y} + x\,.$$

This equation can be solved for y given x, even when B is singular (as is true for real capital structures), along the lines established in ten Raa (1983). The present choice of spaces also facilitates treatment of investment lead times. This is done by simple extension of $B\delta$ beyond the origin of time. The equation and the calculus remain essentially unaltered. For details see ten Raa (1983).

4. SPACE

Spatial economics can be cast in the same mould. The appropriate commodity space is $D'(R^2,R^n)$, consisting of n-vector distributions over the plane. Similarly, the technology space is now $D'(R^2,R^{nxn})$. A simple example is constituted by a spatialisation of the Keynesian consumption equation. Then n=1 (the national pie), $y \in D'(R^2,R)$) is the national product, $c \in D'(R^2,R)$ a spatial propensity to consume (describing the expenditure distribution of one dollar income), and $x \in D'(R^2,R)$ represents non-consumption demand. The equation equilibrium is

$$y = c*y + x.$$

For details see ten Raa (1984).

5. SPACE–TIME

This section addresses the delicate issue of mathematical space selection for a dynamic spatial economy. Such an economy combines dynamic and spatial elements such as the described investment and consumption terms. Perhaps the most natural commodity space to embed those elements in is $D'(RxR^2,R^n)$ which consists of n-vector distributions over time and space jointly. However, often one traces a spatial economy, considered as a whole, through time. This view is especially useful when studying initial value problems for spatial economies, e.g. the ones formulated in Puu (1982). Then such problems can be solved as if they were textbook initial value problems; the only modification is that values do not lie in the reals but in the space of spatial distributions. In this case one takes the alternative commodity space of distributions over time with values in the space of spatial distributions.

A n-vector (spatial) distribution valued distribution (over time) A is a linear continuous functional from the test functions on time, $\phi \in D(R)$, to the n-vector distributions over space, $A(\phi) \in D'(R^2,R^n)$. (Test functions are defined to be infinitely differentiable and to have compact support.) The linearity and continuity conditions are captured elegantly by the following formal definition: A: $D(R) \rightarrow D'(R^2,R^n)$ is a *distribution valued distribution* if $\phi \mapsto \langle A(\phi), \psi \rangle$ is a distribution for all $\psi \in D(R^2)$.

Summing up, we take as the commodity space either $D'(RxR^2,R^n)$, consisting of n-vector distributions over time-space or $L[D(R),D'(R^2,R^n)]$, consisting of n-vector spatial distribution valued distribution over time. The choice is a matter of convenience.

Is the choise of mathematical commodity space a pure matter of convenience, i.e. otherwise immaterial? Yes, the choice can be made on purely opportunistic grounds. The justification of this proposition lies in a deep theorem which states that the space of distributions over time-space and the space of spatial distribution valued distributions over time are essentially the same. More precisely, by the Schwartz (1953–54) kernel theorem there is a bijection between
$A \in L[D(R),D'(R^2,R^n)]$ and (its kernel) $a \in D'(RxR^2,R^n)$. a is obviously defined for separable test functions on time-space, say ϕ x ψ where x is the direct tensor product: $\langle a,\phi x \psi \rangle = \langle A(\phi),\psi \rangle$. The deepness of the theorem lies in the extension of a to all test functions on time-space).

As before, the casting of technology is much the same. For technologies we take either $A \in L[D(R),D'(R^2,R^{nxn})]$ or $A \in D'(RxR^2,R^{nxn})$.

6. APPLICATION

To illustrate the use of our commodity framework for specific models we shall now briefly discuss the application to the trade cycle model of Puu (1982). Detailed analysis would go beyond the scope of the present paper.

Puu studies local income Y and local net export X as functions of time t and location in space, denoted by Euclidean coordinates x and y. He regards X and Y as deviations from equilibrium. Puu assumes that income adjusts in proportion to the degree savings fall short of net export:

$$\dot{Y} = \lambda(X - \sigma Y),$$

where σ is the savings quote, λ denotes adjustment speed and dot time differentiation. He notes that it is usual to relate net exports to income 'abroad' relative to local income. Relative income 'abroad' is measured by the 'curvature' of Y, that is $\partial^2Y/\partial x^2 + \partial^2Y/\partial y^2$ or the Laplacean ΔY. Assuming an import propensity μ and an adjustment process with the same delay as above, Puu obtains

$$\dot{X} = \lambda(\mu\Delta Y - X).$$

The model is reduced by elimination of X:

$$\ddot{Y} + \lambda(1 + \sigma)\dot{Y} + \lambda^2\sigma Y = \lambda^2\mu\Delta Y.$$

This is Puu's equation of a dynamic spatial economy. The initial value conditions are

$$Y(x,y,0) = Y_0(x,y) \quad \text{and} \quad \dot{Y}(x,y,0) = Y_1(x,y).$$

Now we consider the unknown Y as a distribution over time (with spatial distribution values) and incorporate the initial value conditions in the equation by going to HY where H is the Heaviside function (zero on the negatives and one on the positives). Then HY can be shown to fulfill

$$(HY)^{\cdot\cdot} + \lambda(1+\sigma)(HY)^{\cdot} + \lambda^2\sigma HY = \lambda^2\mu\Delta(HY) + [\lambda(1+\sigma)Y_0 + Y_1]\,\delta + Y_0\dot{\delta}.$$

This is a second order differential equation in HY.

Reconsidering HY as a distribution over time-space by the Schwartz kernel theorem and letting E be the fundamental solution of the differential operator we obtain by convolution through E,

$$HY = [\lambda(1 + \sigma)Y_0 + Y_1] * E + Y_0 * \dot{E},$$

where * denotes the convolution product with respect to space.

This is the formal solution of the initial value problem. For the concepts involved we refer the Schwartz (1957). The main task which remains to be done is substantiation of E, but that will not be undertaken here.

7. SUMMARY AND CONCLUSION

The economy is much like an onion (figure 1).

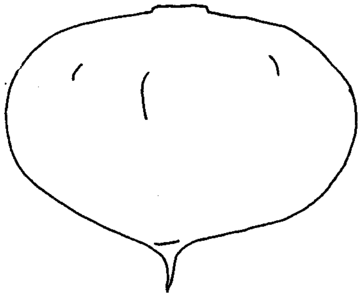

Figure 1

Time and space are treated by dating and locating the commodities. The onion is peeled and the layers are cut. The result is a mashed onion (figure 2).

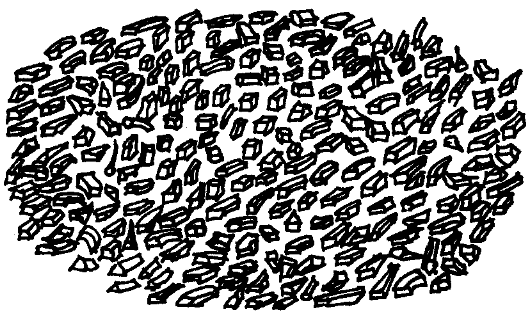

Figure 2

This paper has developed alternative means of analysis. The onion is respected as a full distribution over time and/or space. For dynamic analysis we make a spatial cut and recognise a distribution over time (figure 3).

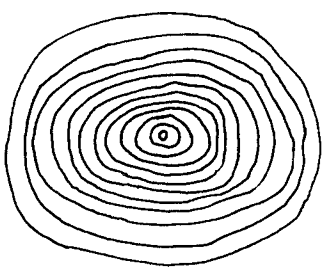

Figure 3

For spatial analysis we separate one layer from the other and obtain a nice spatial distribution (figure 4).

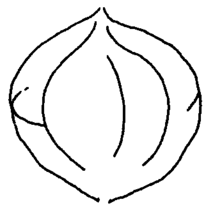

Figure 4

Analysing the dynamics of a spatial system we may adopt alternative points of view. One is to integrate figures 3 and 4 into an onion distribution over time and space jointly (figure 5).

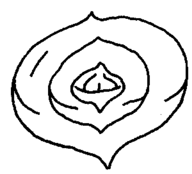

Figure 5

The other point of view is closer to the dynamics at hand. It facilitates analysis of the distribution over time of the spatial onion layers (figure 6).

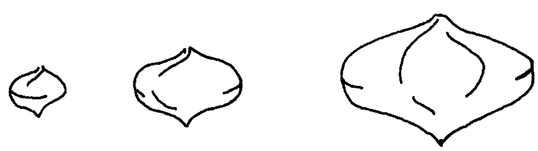

Figure 6

The two points of view on the onion (figures 5 and 6) are equivalent by the Schwartz kernel theorem. This allows opportunistic use of the alternative commodity spaces. For example, the space of time-space distributions (figure 5) is useful for the determination of so-called elementary solutions of particular nonhomogeneous equations, while the space of spatial distribution valued distributions over time (figure 6) is appropriate for handling initial value conditions. I plan to solve the initial value problem for Puu's trade cycle in full detail in a subsequent paper. The proof of the onion is in the eating.

REFERENCES

Debreu, G. (1959), *Theory of Value*, Wiley and Sons, New York.

Puu, T. (1982), 'Outline of a Trade Cycle Model in Continuous Space and Time', *Geographical Analysis*, 14, pp. 1-9.

ten Raa, Th. (1983), 'Dynamic Input–Output Analysis with Distributed Activities', IFAC/IFORS Conference on the Modelling and Control of National Economies, Washington.

ten Raa, Th. (1984), 'The Distribution Approach to Spatial Economics', *Journal of Regional Science*.

Schwartz, L. (1953-54), 'Produits Tensoriels Topologiques', *Séminaire Schwartz* 1, Exposition No 11.

Schwartz, L. (1957), *Théorie des Distributions*, Hermann, Paris.