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Lindahl equilibrium and Schweizer's open club model with semipublic goods

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Abstract

In this paper, we extend Schweizer's open club model to clubs with goods that have a semipublic nature rather than a pure public nature. We study limit core allocations, which are those allocations that remain in the core of a replicated economy. An equivalent notion for open clubs with pure public goods was Schweizer's concept of club efficiency under a variable number of economic agents. We show that given certain conditions, the equivalence of limit core allocations and Lindahl equilibria holds for a wide range of open club economies with semipublic club goods. We also show that extension to a more general class of open club economies seems implausible.

Keywords: Clubs; Club efficiency; Lindahl equilibrium; Limit cores

JEL classification: H41; R51; D71

1. Club efficiency and Lindahl pricing

It is well known that the classical Debreu–Scarf convergence of the core and the set of competitive equilibria in a replicated economy with private goods does not extend well to

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economies with public goods under the standard concept of blocking. Indeed, the well known counterexamples to the Edgeworth conjecture demonstrate that, in the absence of crowding, the per capita cost of supplying a given vector of public goods decreases with the number of agents, thus rendering small coalitions relatively impotent. There are, in principle, two basic methods to overcome this difficulty. One consists in switching to alternative equilibrium concepts, thus "blowing up" the set of equilibria in order to match the larger set of core allocations (Mas-Colell, 1980). The other reduces the set of core allocations by allowing for "congestion effects" (Roberts, 1974; Vasil'ev et al., 1995). This paper belongs to the latter category.

Lindahl equilibrium is a well-known solution concept in the general equilibrium theory of public goods, but its competitive basis is shaky because of the mismatch with the core. In this paper, we show that if the public goods are not pure but feature some form of rivalry in terms of opportunity costs, Lindahl pricing within a club with a variable membership base has a firm competitive basis.

We do so in the context of Schweizer's (1983) model of an open club economy. This model assumes that the club has a variable membership base, drawn from an unlimited pool of potential members. The issue of how to partition a given (closed) population of agents in a number of clubs is not addressed. The possible variation of the numbers of consumer amounts to replication of the economy, and an allocation is now called club efficient if it cannot be improved upon under varying membership bases. To explain the concept further, a membership profile with private and club good consumption plans (for each type of agents) is feasible if the consumption plans can be provided with the initial endowment of the club members, and it is club efficient if no other feasible membership profile yields higher utility to all members. Schweizer (1983) showed that a club efficient allocation must be a competitive, Walrasian equilibrium for an economy with public and private goods and that agents whose numbers are variable do not and should not pay for the public good. His results consolidate the limit core theorem and the Henry George theorem, respectively.[2]

One of the problems of the original formulation of Schweizer (1983) is that the use of a pure public good is unrealistic due to the noncrowding hypothesis. In this paper, we try to remedy this particular problem and introduce intermediate types of goods, denoted as "club goods." These club goods can be purely private or purely public or semipublic. We investigate when a club efficient allocation is a Lindahl equilibrium.

In our formulation, crowding does not enter the utility functions directly. The utility of an agent depends exclusively on his or her own consumption of private goods and club goods. The degree of "publicness" of the club goods is determined by the costs of production. A cost function expresses the input requirements of a membership profile (the composition of a club by type of agents) for each level of club goods consumption (possibly

[2] It can be shown that club efficiency is equivalent to the Debreu–Scarf limit core property, at least for economies with purely private goods. An indirect proof can be based on noting that Schweizer (1983) showed equivalence of the Walrasian equilibrium concept and club efficiency. Debreu and Scarf (1963) showed equivalence of Walrasian equilibria and the limit core. Hence, club efficiency, the Walrasian equilibrium concept, and the limit core property are the same.

varying by type of agents). In the polar cases of private and pure public goods, the cost function is linear and constant, respectively.

The main contribution of this paper is the delineation of cost functions of club goods such that a club efficient allocation is a Lindahl equilibrium. One may expect to encounter the membership profile of such a club efficient allocation in an economy with a continuum of agents, not plagued by integer problems. More interesting, the prices supporting a club efficient allocation are Lindahl prices.

The public goods literature incorporates a tricky division as regards the exogeneity or endogeneity of the number of consumers and the level of the public goods. In the older literature, going back to Foley (1970), the number of consumers is fixed and the level of the public goods is variable. However, the public goods are neither pure nor fixed but determined by preferences. Foley defined a Lindahl equilibrium as a set of prices, economy-wide for private goods, and individualized for public goods, such that markets clear. He proceeded to demonstrate that Lindahl equilibria are in the core. Ellickson (1973) showed that a Lindahl equilibrium need no longer be in the core when public goods are not pure but have opportunity costs that increase with the number of consumers; he also showed that the core may even be empty. Convexity (in particular of technology) plays no role in the proof that a Lindahl equilibrium allocation is in the core when crowding is present but does play a role in showing that any core allocation is a Lindahl equilibrium allocation and in showing that the set of core or Lindahl equilibrium allocations is nonempty when crowding is present. We follow Ellickson in admitting nonpure public goods but assume some convexity at the aggregate level of technology to keep scope for positive results.

Milleron (1972) considered a replicated economy with pure public goods. The trouble with pure public goods is that they are not replicated along with the population in the economy and their per capita opportunity costs vanish. To keep the Lindahl equilibria in the core, Milleron changed the preferences or endowments of the consumers as the economy becomes large. Even then, the core does not shrink to the set of Lindahl equilibria. Vasil'ev et al. (1995) were able to let the core shrink to the set of Lindahl equilibria but also had to change the consumers' utility functions as they were replicated. Conley (1994) obtained this result assuming that consumers are either asymptotically satiated or strictly nonsatiated in public goods; these are extreme polar cases of consumer utility functions. We need no such assumptions in the context of the open club model with semipublic club goods.

The roles of consumer numbers and public good levels were reversed in Schweizer (1983). He solved for allocations that included a club membership profile. On the other hand, he fixed the level of the public goods and devised "Lindahlian" price support of club efficient allocations but had to assume that some types of agents are given in fixed numbers. The other types escape taxation as they can bring in more members of their types and thus may spread the burden of their collective contribution to the public good. We follow Schweizer in letting the numbers of consumers be variable, but the public goods are neither pure nor fixed. At least in principle, the use of the open club model may drive the main result that Lindahl equilibria exhaust the core, simply by increasing the set of Lindahl equilibria, but we do not believe so. The Lindahl equilibria we analyze feature not only utility maximizing agents but also profit maximizing club administrators. Members pay their marginal cost. Hence, there are pricing rules for all club goods. The multiplicity of equilibria is no larger than in the Arrow–Debreu model. The main use of the open membership base is that the analysis is not plagued by the integer problem.

We look at the provision of *club goods* that in principle have a *semipublic* nature. It is assumed that these commodities are provided through the club and therefore are principally locally collective. But their rivalry properties might be different from that of a purely local public good. We model this by means of a cost function that associates input requirements with members' demands for these club goods. Our main theorem states that, for certain club goods with a semipublic nature, the notions of club efficiency and Lindahl equilibrium remain equivalent. For this, we extend Schweizer's (1983) equivalence theorem (of Walrasian equilibrium and club efficiency) to a model in which the aggregation function for the club goods has a certain specification and certain properties. We also show that it cannot be expected that our Lindahl equivalence result can be extended further to more general specifications of the aggregation function.

The second section develops the model, Section 3 states and proves our equivalence result, and Section 4 concludes the paper with a discussion of the result, its relationship to the literature, and its implications.

2. Clubs and semipublic club goods

In this section, we introduce a model of an open club economy consisting of a membership base, an allocation of private goods consumed, and an allocation of so-called club goods, which are provided collectively. The membership base as part of our model of a club represents the "openness" of the club. In our theory, we use a club as a replication device.

We consider an economy with a finite set of consumer types denoted by $t=1, \ldots, T$. A vector $n \in \mathbb{R}_+^T$ represents a *profile* of a coalition of economic agents, comprising n^t members of type t. A profile $n \in \mathbb{R}_+^T$ forms the membership base of the club economy. Throughout, we assume that agents of the same type are treated equally, i.e., agents of the same type consume the same quantities of private as well as club goods. This assumption enables us to discuss replication properly.[3]

We consider a situation with $\ell \in \mathbb{N}$ private goods. Agents of type t are endowed with a commodity bundle $w^t \in \mathbb{R}_+^\ell$. It is assumed that $w^t > 0$ for all t.[4] Private consumption of an agent of type t is now given by $x^t + w^t \in \mathbb{R}^\ell$, where x^t denotes the net consumption of type t. A net consumption plan is now a vector of net consumption bundles $x = \left(x^1, \ldots, x^T\right) \in \mathbb{R}^{\ell T}$. Total net consumption of private goods in a club with membership base $n \in \mathbb{R}_+^T$ is represented by $\overline{x} = \left(n^1 x^1, \ldots, n^T x^T\right) \in \mathbb{R}^{\ell T}$.

[3] In the standard model of a replicated pure exchange economy, the equal treatment property can be shown to hold if preferences are strictly convex (Debreu and Scarf, 1963).

[4] Here, we define $w^t > 0$ if $w^t = 0$ and $w^t \neq 0$.

There are $m \in \mathbb{N}$ club goods. Each club good is provided collectively by the club to its members. Again assuming equal treatment, an agent of type t now consumes the club goods at levels given by a vector $y^t \in \mathbb{R}_+^m$. The consumption plan for club goods is represented by the vector $y = (y^1,\ldots,y^T) \in \mathbb{R}_+^{mT}$. Total consumption of club goods in a club with membership base $n \in \mathbb{R}_+^T$ is now represented by $\overline{y} = (n^1 y^1,\ldots,n^T y^T) \in \mathbb{R}_+^{mT}$. The premise of this paper is that the total consumption of club goods (by type) determines cost. Cost must be a function of the product of population and the bundle consumed by each type. This functional form specification paves the way for the competitive foundation of Lindahl prices. This is formalized as follows.

Modelling hypothesis. The production technology is represented by the induced cost function C: $\mathbb{R}_+^{mT} \to \mathbb{R}_+^{\ell}$ which for every membership base $n \in \mathbb{R}_+^T$ and consumption plan $y \in \mathbb{R}_+^{mT}$ assigns to the total consumption bundle $\overline{y} = (n^1 y^1,\ldots,n^T y^T) \in \mathbb{R}_+^{mT}$ a bundle of private goods $C(\overline{y}) \in \mathbb{R}_+^{\ell}$ that is used to create the club goods at these levels.[5]

The modelling hypothesis equates the marginal cost of a member with the marginal cost of his or her club bundle. Hence, entry fees or subsidies depend only on the consumption bundle of a particular type. This is the quintessence of Lindahl prices and explains why they can support a club efficient allocation.

This framework, however, encompasses a number of interesting cases. The club goods have a purely private nature if $C(\bar{y}) = \tilde{C}(\sum_{t=1}^{T} n^t y^t)$, where the cost function $\tilde{C}: \mathbb{R}_+^m \to \mathbb{R}_+^{\ell}$ represents a standard private goods production technology converting the ℓ private good inputs into m private good outputs. (This reduces the model to the standard setting of a pure exchange economy).

Second, the club goods have a *purely public* nature in the sense of Schweizer (1983) if $C(\overline{y}) = Z \in \mathbb{R}_+^{\ell}$ for every $\overline{y} \in \mathbb{R}_+^{mT}$, where Z is some fixed input vector.

Finally, there are many intermediate possibilities, giving the club goods a semipublic nature. For example, if $C(\bar{y}) = \tilde{C}(\max_{t=1,\ldots,T} n^t y^t)$, where the max operator on $\mathbb{R}^m$ is defined by $\max_i(y^1, y^2) = \max(y_i^1, y_i^2)$, $i=1,\ldots,m$, and, as before, $\tilde{C}: \mathbb{R}_+^m \to \mathbb{R}_+^{\ell}$ represents a standard private goods production technology, we can interpret the club goods to be based on a fixed infrastructure such as a network. The capacity of the network has to handle the peak demands, which in turn determines the construction costs. A contemporary example of such a situation is that of the provision of access to Internet through a so-called "Internet Service Provider" (ISP). One can interpret an ISP as a club that provides access to Internet services to their members. The cost function $\tilde{C}$ introduced here exactly represents the cost structure for such an ISP. Capacity of the ISP's server needs to be based on peak demands for Internet access at the different time moments during a standard period of time. These time moments can be represented by the discrete parameter t.

These examples feature an important commonality, namely, convexity. In the purely public case in the sense of Schweizer, the induced cost function C is constant, which is obviously convex. In the purely private and semipublic cases, C is induced by a private goods cost function $\tilde{C}$. If $\tilde{C}$ is convex, as is standard in neoclassical production theory (excluding increasing returns to scale in production), then so is C in either case, as the

[5] We may allow substitution of inputs by generalizing C to a correspondence.

latter is the composition of $\tilde{C}$ and either summation (of private goods) or maximization (of semipublic goods).

A *club* is now introduced as a tuple $(n^t, x^t, y^t)_{t=1,\ldots,T}$, where $n=(n^1,\ldots,n^T)\in\mathbb{R}^T_+$ is a profile, $x=(x^1,\ldots,x^T)\in\mathbb{R}^{\ell T}$ a net private consumption plan, and $y=(y^1,\ldots,y^T)\in\mathbb{R}^{mT}_+$ is a club good consumption plan. A club $(n^t, x^t, y^t)_{t=1,\ldots,T}$ is *feasible* if

$$\sum_{t=1}^{T} n^t x^t + C\left(n^1 y^1,\ldots,n^T y^T\right) \leqq 0. \tag{1}$$

Net demands for the private goods and the costs for the provision of the club goods sum to zero at most.[6] For simplicity, there is no production of private goods. Its inclusion would be a straightforward extension of the model.

A consumer of type t has an extended utility function $U^t:\mathbb{R}\times\mathbb{R}^m\rightarrow\mathbb{R}$ over his total private and club good consumption. However, since his initial endowment w^t is fixed, we may simply write $U^t(x^t, y^t)$. In principle, we allow an agent to have short positions in all commodities.

Next, we introduce our main efficiency concept. Consider two feasible clubs given by $(n^t, x^t, y^t)_{t=1,\ldots,T}$ and $(n_0^t, x_0^t, y_0^t)_{t=1,\ldots,T}$. The club $(n^t, x^t, y^t)_{t=1,\ldots,T}$ is an *improvement* over the club $(n_0^t, x_0^t, y_0^t)_{t=1,\ldots,T}$ if

$$U^t\left(x^t,y^t\right) > U^t\left(x_0^t,y_0^t\right) \text{ for every } t \text{ with } n^t>0.$$

Following Schweizer (1983), if no such improvement exists for a club $(n_0^t, x_0^t, y_0^t)_{t=1,\ldots,T}$, then $(n_0^t, x_0^t, y_0^t)_{t=1,\ldots,T}$ is called *club efficient*.

A feasible club $(n_0^t, x_0^t, y_0^t)_{t=,\ldots,T}$ is a *Lindahl equilibrium* if there exist a private goods price vector $p\in\mathbb{R}^\ell_+$ and personalized admission price vectors $p^1,\ldots,p^T\in\mathbb{R}^m_+$ such that the following conditions are satisfied:

(i) For every $t\in\{1,\ldots,T\}$ with $n_0^t>0$, the allocation satisfies the consumer utility maximization condition

$$\left(x_0^t,y_0^t\right) = \operatorname{argmax} U^t\left(x^t,y^t\right) \text{ subject to } px^t + p^t y^t \leqq 0.$$

(ii) The club $(n_0^t, x_0^t, y_0^t)_{t=,\ldots,T}$ satisfies a budget balance condition, i.e.,

$$\sum_{t=1}^{T} n_0^t p^t y_0^t = pC\left(n_0^1 y_0^1,\ldots,n_0^T y_0^T\right).$$

(iii) The club $(n_0^t, x_0^t, y_0^t)_{t=,\ldots,T}$ is optimal in the sense that, for every alternative club $(n^t, x^t, y^t)_{t=1,\ldots,T}$

$$\sum_{t=1}^{T} n^t p^t y^t \leqq pC\left(n^1 y^1,\ldots,n^T y^T\right).$$

[6] We remark that Schweizer (1983) introduces a given endowment for the club, denoted by $F\geqq 0$, that covers the provision costs of the public goods and the net demands for private goods. In that case, in Eq. (1), the zero is replaced by F. Here, we limit our discussion to the case without such an endowment.

By the first condition, consumers maximize their utility given the market prices for the private goods and the personal admission prices for the semipublic club goods. The fees collected cover the costs of the provision of the club goods by the second condition. The third condition stipulates that a public administration is in charge of the provision of the club goods and admission prices and as such has the objective to maximize its "profits" (this maximal profit is zero by the second condition). This condition is not included here because we consider the number of consumers to be exogenous (see Foley, 1970 and other papers referenced in Section 1). However, since our theorem will entail that club efficiency implies Lindahl pricing, the result is only strengthened by the inclusion of the third condition in the definition of Lindahl equilibrium.

3. A decentralization result

Relatively, little is assumed to arrive at complete decentralization of efficient clubs through appropriate price systems. Following Foley (1970) and Schweizer (1983), positivity of prices is ensured to render a complete decentralization through Lindahl pricing.

Axiom.

- **(a)** For every type t=1, . . ., T the utility function U^t is assumed to be continuous, quasi-concave, and strongly monotonic.
- **(b)** The club good production technology is convex in the sense that the cost function C: $R_+^{mT} \rightarrow R_+^{\ell}$ is convex.

In the context of this assumption, we have the following result.

Theorem. *Under the properties stated in the Axiom, every efficient club $(n_0^t, x_0^t, y_0^t)_{t=1,...,T}$ with a strictly positive endowment, $\sum_{t=1} n_0^t w^t \gg 0$, can be supported as a Lindahl equilibrium with strictly positive prices.*

Proof. Let the club $(n_0^t, x_0^t, y_0^t)_{t=1,...,T}$ be efficient.

We construct the following sets. First, for every $t \in T$, we define the preferred set,

$$B^t = \left\{ \left(x^t, 0,..., 0, y^t, 0,..., 0\right) | U^t\left(x^t, y^t\right) > U^t\left(x_0^t, y_0^t\right) \right\} \subset \mathbb{R}^{\ell} \times \mathbb{R}_+^{mT}.$$

In this definition, we let y^t be at location 1+t.

Now for any profile $n \in \mathbb{R}_+^T$, we define the preferred set,

$$B_n = \sum_{t=1}^{T} n^t B^t = \left\{ \left(\sum_{t=1}^{T} n^t x^t, n^1 y^1,..., n^T y^T \right) \middle| U^t(x^t, y^t) > U^t(x_0^t, y_0^t) \text{ for all } t \right\}.$$

Finally, we let[7]

$$B = \cup\left\{B_n | n\in\mathbb{R}^T_+ \text{ such that } n>0\right\} \subset \mathbb{R}^{\ell} \times \mathbb{R}^{mT}_+.$$

Second, we introduce the feasible set,

$$D = \left\{ \left(-C(n^1y^1,\ldots,n^Ty^T) - z, n^1y^1,\ldots,n^Ty^T \right) \middle| \begin{array}{l} n>0,\ z\in\mathbb{R}^{\ell}_+, \\ y^1,\ldots,y^T\in\mathbb{R}^m_+ \end{array} \right\}.$$

We remark that also $D\subset\mathbb{R}^{\ell+mT}$.

B^t is convex by quasi-concavity of U^t for every type t. It follows that B_n is convex for each n. Because $\lambda B_n+(1-\lambda)\ B_{\hat{n}}=B_{\lambda n+(1+\lambda)\hat{n}}$ for $\lambda \in [0,1]$, it follows that the set B is convex. Furthermore, from continuity of U^t for every type t, the set B is open in $\mathbb{R}^{\ell} \times \mathbb{R}^{mT}_+$.

We show that D is convex. Let $(y^1, \ldots, y^T, z, n)$ and $(\hat{y}^1, \ldots, \hat{y}^T, \hat{z}, \hat{n})$ constitute (but not be) members of D. Define $v=(n^1 y^1, \ldots, n^T y^T)$ and $\hat{v}=(\hat{n}^1 \hat{y}^1,\ldots, \hat{n}^T \hat{y}^T)$. Then $(-C(v)-z, v) \in D$ as well as $(-C(\hat{v})-\hat{z}, \hat{v})\in D$.

Now consider $\lambda \in [0, 1]$. We have to show that there exists a tuple $(\tilde{y}^1,\ldots, \tilde{y}^T, \tilde{z}, \tilde{n})$ such that $(-C(\tilde{v})-\tilde{z}, \tilde{v})\in D$ where $\tilde{v}=(\tilde{n}^1\tilde{y}^1,\ldots, \tilde{n}^T\tilde{y}^T)\ \tilde{v}=\lambda v+(1-\lambda)\hat{v}$ and $C(\tilde{v})+\tilde{z}=\lambda(C(v)+z)+(1-\lambda)(C(\hat{v})+\hat{z})$. This can be accomplished by selecting $\tilde{y}^t=\lambda n^t y^t+(1-\lambda)\hat{n}^t \hat{y}^t$ for every t, $\tilde{n}^t=1$, and

$$\tilde{z} = \lambda C(v) + (1-\lambda)C(\hat{v}) - C(\tilde{v}) + \lambda z + (1-\lambda)\hat{z}.$$

Now, $\tilde{v}=\lambda v+(1-\lambda)\hat{v}$ and by convexity of the cost function C, it follows that

$$\tilde{z} = \lambda C(v) + (1-\lambda)C(\hat{v}) - C(\tilde{v}) + \lambda z + (1-\lambda)\hat{z} \geqq C(\lambda v + (1-\lambda)\hat{v}) - C(\tilde{v}) + \lambda z + (1-\lambda)\hat{z} = \lambda z + (1-\lambda)\hat{z}.$$

Hence, $\tilde{z}\geqq 0$ and thus indeed $(-C(\tilde{v})-\tilde{z}, \tilde{v}) \in D$, finishing the proof that D is convex.

We define the cone generated by the feasible set D by

$$\overline{D} = \{\lambda d | d\in D \text{ and } \lambda\geqq 0\}.$$

By convexity of D, it follows that $\overline{D}$ is a convex cone.

We claim that B and $\overline{D}$ do not intersect. Suppose to the contrary that $(n^t, x^t, y^t)_{t=1,\ldots,T}$ constitutes a member of B, $(\hat{n}^t, \hat{y}^t)_{t=1,\ldots,T}$, and $\hat{z}\in\mathbb{R}^{\ell}_+$ constitute a member of D, and $\lambda\geqq 0$ such that

$$\left(\sum_{t=1}^{T} n^t x^t, n^1y^1,\ldots,n^Ty^T\right) = \left(-\lambda C\left(\hat{n}^1\hat{y}^1,\ldots,\hat{n}^T\hat{y}^T\right) - \lambda\hat{z}, \lambda\hat{n}^1\hat{y}^1,\ldots,\lambda\hat{n}^T\hat{y}^T \right).$$

If $\lambda>0$, it follows that $\hat{n}^t\hat{y}^t = \frac{n^t}{\lambda}y^t$ and that

$$\sum_{t=1}^{T}\frac{n^t}{\lambda}x^t = -C\left(\frac{n^1}{\lambda}y^1,\ldots,\frac{n^T}{\lambda}y^T\right) - \hat{z} \leqq -C\left(\frac{n^1}{\lambda}y^1,\ldots,\frac{n^T}{\lambda}y^T\right).$$

This implies that the club $(n^t/\lambda, x^t, y^t)_{t=1,\ldots,T}$ is feasible and improves upon the club $(n^t_0, x^t_0, y^t_0)_{t=1,\ldots,T}$. Since this contradicts the efficiency hypothesis, it follows that $\lambda=0$ and the only conceivable intersection point of B and $\overline{D}$ is the origin. However, since B is open in

[7] See footnote 3 for the vector inequality notation.

$\mathbb{R}^{\ell} \times \mathbb{R}_{+}^{mT}$, a perturbation of the origin to the left, with the first (ℓ-dimensional) component slightly negative, would still belong to B. By construction of D and $\overline{\mathrm{D}}$, the perturbation would also belong to $\overline{\mathrm{D}}$, contradicting that the origin is the only conceivable intersection point. Hence, the intersection is empty.

By the separating hyperplane theorem and the fact that $\overline{\mathrm{D}}$ is a cone, there exist $p \in \mathbb{R}_{+}^{\ell}$ and $p^1,\ldots,p^T \in \mathbb{R}_{+}^{m}$ not all equal to zero such that

$$\left(p, p^1, \ldots, p^T\right) B \geqq 0 \geqq \left(p, p^1, \ldots, p^T\right) \overline{D}. \tag{2}$$

By strong monotonicity of U^t, it can be concluded that B is comprehensive, and therefore, $(p, p^1, \ldots, p^T) > 0$. It must value $(-C(v), v) \in D$ nonpositively: $(p^1, \ldots, p^T)v \leqq pC(v)$ for all $v \geqq 0$. Since $p=0$ would imply $(p, p^1, \ldots, p^T)=0$, we must have $p>0$. Also, by assumption that the aggregated total endowment is strictly positive, we may conclude that $\sum n_0^t pw^t > 0$. Thus, there is a type t with $n_0^t > 0$ and $pw^t > 0$. For this type t, an interior consumption plan is feasible with respect to $px^t + p^t y^t \leqq 0$. Hence, by strong monotonicity and continuity of U^t, using a standard argument, $p \gg 0$ as well as $p^t \gg 0$. Hence, by nonzero endowment assumption, $pw^t > 0$ for all t. By the same argument, all $p^t \gg 0$. We will now prove that these prices constitute a Lindahl equilibrium.

First, we show the consumer's utility maximization condition. Suppose that the tuple given by $(x^t, 0, \ldots, 0, y^t, 0, \ldots, 0)$ —with y^t at location $1+t$—satisfies $U^t(x^t, y^t) > U^t(x_0^t, y_0^t)$. In fact, since $p \gg 0$, $pw^t > 0$, and the utility function is strongly monotonic and continuous, the same holds for a pair of slightly smaller vectors. Now, from the separation property (2) and the strict positivity of all prices, it is concluded that $px^t + p^t y^t > 0$.

It remains to show that (x_0^t, y_0^t) satisfies the budget condition $px_0^t + p^t y_0^t = 0$ if $n_0^t > 0$. Indeed, from the feasibility condition for $(n_0^t, x_0^t, y_0^t)_{t=1,\ldots,T}$, it follows that there is some $z \in \mathbb{R}_{+}^{\ell}$ such that

$$\left(\sum_{t=1}^{T} n_0^t x_0^t, n_0^1 y_0^1, \ldots, n_0^T y_0^T\right) = \left(-C\left(n_0^1 y_0^1, \ldots, n_0^T y_0^T\right) - z, n_0^1 y_0^1, \ldots, n_0^T y_0^T\right) \in D.$$

From the separation property (2) it then follows that

$$\sum_{t=1}^{T} n_0^t p x_0^t + \sum_{t=1}^{T} n_0^t p^t y_0^t = \sum_{t=1}^{T} n_0^t \left(px_0^t + p^t y_0^t\right) \leqq 0. \tag{3}$$

By strong monotonicity, $(x_0^t, 0, \ldots, 0, y_0^t, 0, \ldots, 0)$ belongs to the boundary of $B^t \subset B$. From Eq. (2), it immediately follows that $px_0^t + pty_0^t = 0$. Hence, each term in Eq. (3) must be zero. Since $n_0^t = 0$ for all types t, it now immediately can be concluded that $px_0^t + p^t y_0^t = 0$ if $n_0^t > 0$.

Together with previously shown statement, this proves that (x_0^t, y_0^t) indeed solves the consumer's problem if $n_0^t > 0$.

Second, we consider the financial balance condition. Since, as shown above, each term in Eq. (3) must be zero, it follows immediately that

$$\sum_{t=1}^{T} n_0^t p^t y_0^t = -\sum_{t=1}^{T} n_0^t p x_0^t = pC\left(n_0^1, y_0^1, \ldots, n_0^T y_0^T\right), \tag{4}$$

where the last equality reflects the fact that the feasibility constraint is binding, using strong monotonicity.

Finally, we consider the problem of the public administration. Since the prices value D nonpositively, we have that

$$\sum_{t=1}^{T} n^t p^t y^t - p\ C\left(n^1 y^1, \ldots, n^T y^T\right) \leqq 0.$$

This proves that $(y_0^1, \ldots, y_0^T, n_0)$ indeed solves the public administration's problem. This completes the proof of the theorem. □

The converse of the theorem also holds. The proof is an easy adaptation of Schweizer's (1983) proof of his second theorem. Thus, we have a true equivalence result.

The implementation of more general club good cost functions is probably very hard, if not impossible. In the next example, we consider a cost function that is more general but fails to lead to equivalence of the set of efficient clubs and the set of Lindahl equilibria. Semipublic goods, as we defined them, have a distinct structure in that only total consumption by type $\bar{y} = \left(n^1 y^1, \ldots, n^T y^T\right) \in \mathbb{R}_+^{mT}$ affects their provision. In general, a club with profile n and club goods demands y may impose resource requirements in a way that is not separable by type.

Counterexample. Consider an economy setting with one private and one club good, i.e., $\ell = m = 1$, and two types of consumers, i.e., T=2, with the following utility functions:

$$U^1(x, y) = \min(2x + 4, y);$$

$$U^2(x, y) = \min(2x + 3, 2y).$$

Now consider a production structure for the club good that does not satisfy the functional form considered in our model. The cost function is given by

$$C(n^1, n^2, y^1, y^2) = \max_{t \in \{1,2\}} n^t \max_{t \in \{1,2\}} y^t .$$

This cost function can be interpreted as representing a semipublic good of which the provision is based on the maximal consumption capacity requested, where the maximal capacity is max n^t. This cost function is convex, but here, costs are not a function of the total consumption of club goods by type, $n^1 y^1, \ldots, n^T y^T$. The trade-off within types between members and mean consumption does not hold. Total consumption of the club goods by type is shown to be an insufficient statistic for core equivalence.

Consider the club given by n_0=(1, 1), $x_0^1 = x_0^2 = -1$, and $y_0^1 = y_0^2 = 2$. This club is efficient, as we demonstrate first.

We show that U^2 cannot be lifted over its club level, 1, whenever $n^2 > 0$, $U^1 \geqq 2$, and feasibility is fulfilled. Invoking linear homogeneity with respect to n, feasibility now requires

$$n^1 x^1 + x^2 + \max\left(n^1, 1\right) \cdot \max\left(y^1, y^2\right) \leqq 0.$$

Hence,

$$x^2 \leqq -n^1 x^1 - \max\left(n^1, 1\right) \cdot y^1 .$$

Substituting $x^1 \geqq -1$ and $y^1 \geqq 2$ (both from $U^1 \geqq 2$) obtains

$$x^2 \leqq n^1 - 2\max\left(n^1, 1\right) \leqq -1.$$

Hence, $U^2(x^2, y^2) \leqq 1$, proving club efficiency.

This utility level is obtained only if $n^1=1$ and the feasibility constraint is binding:

$$x_0^1 + x_0^2 + \max(y_0^1, y_0^2) = 0.$$

Now, suppose that there is a Lindahl price p. Substituting the Lindahl break-even constraint for the semipublic goods, the sum of the consumers' budgets is zero. Since each of these budgets is nonpositive, they are all zero. Better clubs must be priced higher, hence positively. But this is not so. Indeed, consider any club with n arbitrary and $(x^1, y^1)=(1/2, 1)$. Now $(x^2, y^2)=(-1/2, 1)$, and therefore, a consumer of type 2 prefers this club to the original one. This consumption bundle is half of the club-efficient bundle, $(x_0^2, y_0^2)=(-1, 2)$, which has zero value. This implies that it is affordable. This in turn implies that the efficient club cannot be supported as a Lindahl equilibrium.

4. Discussion

Our theorem provides price support to club allocations that cannot be improved upon. These prices are linear, unlike Mas-Colell's (1980) personalized price schedules—extended to economies with multiple private goods by Diamantaras and Gilles (1996) and to club economies by Gilles and Scotchmer (1997)—or the admission fees or "wages" used by Barham and Wooders (1998). The theorem and its proof are adaptations of Schweizer's (1983) theorem on club efficient allocations. He obtains the Henry George Theorem for economies with fixed public goods and associated inputs and, if the latter are zero, the welfare and core limit theorems. In the present paper, club goods are not exogenous but endogenous, namely, the outcome of competition among utility maximizers. Moreover, in principle, these club goods are not purely public but semipublic.

It is well known that there is no competitive basis for Lindahl equilibria in pure public goods economies (Milleron, 1972; Bewley, 1981). Wooders (1978) has conjectured that the core shrinks when there is crowding, but Conley and Wooders (1997) show that the second welfare theorem is generally false. Barham and Wooders (1998) provide useful relationships between optima and competitive equilibria, but all these papers concern economies with only one private and one public good. In these papers, the private good required to provide n members with y units of the public good is given by $C(n, y)$ and utility features a congestion argument represented by $U(x, y, n)$. Now, the reduced form is given by $U(x-C(n, y)/n, y, n)$. Wooders (1978, p. 336) assumes that the best value with respect to y is maximized further for two consecutive integer values of n. In other words, the expression (maximized with respect to y) is assumed locally constant in n. This constitutes a knife-edge, joint assumption on C and

U. Now, in this paper, we have essentially absorbed the (utility) congestion argument in the costs. Denoting the resulting cost and utility functions by $\tilde{C}$ and $\tilde{U}$, respectively, the relation becomes $\tilde{U}(x-\tilde{C}(n, y)/n, y, n)$=$\tilde{U}(x-\tilde{C}(n, y)/n, y)$. By the envelop theorem, the maximum with respect to y is locally constant with respect to n if $\tilde{C}(n, y)/n)$ is locally constant with respect to n. This implies $\tilde{C}(n, y)=n \cdot c(y)$. Our modelling hypothesis, however, is $\tilde{C}(n, y)=C(ny)$. Wooders' and our approaches are consistent if the per capita cost function c (which includes the congestion costs) features constant returns to scale.

For economies with multiple private and public goods, Conley (1994) conjectures that the core of a public goods economy converges only in the knife-edge case in which the increasing returns to coalitional size are precisely offset by crowding, diminishing marginal returns in production, or something similar. In a sense, we have articulated this intuition. For example, if the public goods function is $C(ny)=F+(ny)^2$ (everything scalar), then club efficiency brings about the efficient scale of production, $n_0 y_0 = \sqrt{F}$, an argument that extends to more general production possibilities.

An alternative model of an economy with multiple public goods such that the Lin-dahl equilibrium emerges, has been undertaken by Vasil'ev et al. (1995). That paper uses an alternative core concept based on utility levels of members of blocking coalitions depending on the replica size and the coalition structure. The comparison is as with Wooders et al., without the congestion argument in the costs and with nT-dimensional (the number of types). For one type, the reduced form reads $U(x-C(y)/n, y, n)$, and we may absorb the (third) congestion argument in the costs. Although our approach to club goods may seem different, the two approaches are closely related, in the sense that the opportunity cost of individual public—or club—goods consumption is not reduced with the size of the economy in either model. From this perspective, the contribution of our paper is a demonstration that Schweizer's theorem encompasses the core limit theorem of Vasil'ev et al. (1995).

The just mentioned replication literature has attempted to provide a competitive basis for Lindahl equilibria by modelling congestion on the demand side, while we have put congestion on the supply side. In a way, this is a return to the intuition of Ellickson (1973, p. 417): what matters is the convexity of the aggregate technology set. When the number of consumers varies freely, the convexity ensures that any core allocation is a Lindahl equilibrium, provided that cost is a function of the product of the subpopulation of each type and the club bundle they consume. Then, Lindahl prices also represent the marginal effect of adding another person of a given type to the club. This explains when and why Lindahl equilibria have a competitive basis in economies with semipublic goods.

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