A Neoclassical Analysis of TFP Using Input-Output Prices

Thijs ten Raa

Abstract

Input-output analysis and neoclassical economics do not seem to mix. Neoclassical economists consider input-output analysis a futile exercise in central planning or at least resent the separation between the quantity and value systems. Conversely, input-output economists resent marginal analysis without an understanding of the underlying structure of the economy. In this paper I put the perceptions upside down, by analyzing productivity. I ground the concept in the orthodox neoclassical general equilibrium framework. Then I introduce a linear specification and use input-output analysis to derive a measure of total factor productivity without using value shares of factor inputs. In other words, input-output analysis has the potential to explain prices which neoclassical growth accountants take at face value.

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1. Introduction

During one of our very last discussions, Wassily Leontief asked me: "What are you doing these days?" I replied that I reconcile input-output analysis and neoclassical economics. He leant back, thought, looked me straight into the eyes, and said "Should be easy."

Yet input-output analysis and neoclassical economics seem hard to mix. The resentment between the two schools of economics is a two-way affair.

Neoclassical economists consider input-output analysis a futile exercise in central planning. The relationship between the delivery of a bill of final goods and its requirements in terms of gross output and factor inputs is considered mechanical, with no or little attention paid to the role of the price mechanism in the choice of techniques (Leontief, 1941). True, input-output analysis is used to relate prices to factor costs, but here too the analysis is considered mechanical as input-output coefficients are presumed to be fixed. To make things worse, the quantity and value analyses are perceived to be disjunct, with no interaction between supply and demand.
Conversely, input-output economists consider neoclassical economics a futile exercise in marginal analysis that fails to grasp the underlying structure of the economy. Firms supply up to the point that marginal revenue equals marginal cost and set the price accordingly. But does not marginal cost depend on all prices in the system, including the one of the product under consideration? And if the answer is yes, should not we take into account the interindustry relations, i.e. apply input-output analysis?

Many, including myself, have been held captive by these perceptions. Yet they are misleading. Instead of criticizing the critiques, a meta-analysis which is doomed to have little input, I provide some shock therapy, that puts the perceptions upside down, by analyzing a concrete issue, namely productivity measurement. Why productivity? Well, the standard, neoclassical measure of productivity growth, the so called Solow residual between output growth and input growth, employs market values of labor and capital to compute a weighted average of their input growth rates. Now it can be shown that the Solow residual is equal to a weighted average of the growth rates of the real wage and the real rental rate of capital. (In other words, total factor productivity growth is the sum of labor productivity growth and capital productivity growth.) By taking the wage rate and rental rate at market values in computing the Solow residual, neoclassical economists accept at face value what they are supposed to measure.

In this paper I adopt the methodological position of neoclassical economics, by which productivity is defined as the marginal contribution of factors inputs, but apply input-output analysis to determine its value. The analysis is framed in the orthodox general equilibrium model, which subsequently will be specified to accommodate growth accounting. I will recover the neoclassical formulas, such as the Solow residual, but the structure of the economy will be exploited to determine the values.

2. Earlier work

My first attempt to reconcile input-output and neoclassical economics is in the sequel papers ten Raa (1994) and ten Raa and Mohnen (1994). We maximized the value of final demand at world prices. Final demand for non-tradable commodities was simply fixed at the observed level. In short, we expanded final demand for tradable commodities, but not for non-tradable commodities. The model lacks a utility foundation. We rectified this in ten Raa (1995) and Mohnen et al. (1997), where we maximized the level of the entire domestic final demand vector, given its proportions. In ten Raa and Mohnen (2002) we investigate not only the frontier of the economy, but also the fluctuations of the observed economy about its frontier. All the aforementioned papers are about small, open economies with exogenous prices for the tradable commodities. The main contribution of this paper is that it lays out the theory for a closed economy. In other words, we make the step from partial to general equilibrium analysis.

Subsidiary, I now present the theory from an orthodox mathematical economic perspective, say Debreu (1959). First and foremost, the two "practical" approaches of input-output analysis and growth accounting are clearly embedded in a unifying framework. Second, the general equilibrium framework endogenizes the value shares used in growth accounting exercises (such as Jorgenson and Griliches, 1967). Third, the exposition makes Debreu's framework accessible to applied economists.

3. Growth accounting

There are two sources of growth. The first is that economies produce more output, simply because they use more input, such as labor. Of course, this is a mere size effect; there is no increase of the standard of living. The second source of growth is more interesting. Economies produce more output per unit of input, because of technological progress. The classical exposition of these two sources of growth is Solow (1957). He demonstrates that the residual between output and input growth measures the second source of growth, that is the shift of the production possibility frontier. In his analysis Solow makes two assumptions. First, the production function is macro-economic, hence transforming labor and capital into a single output. Second, the economy must be perfectly competitive, so that factor inputs are priced according to their marginal productivities. By the first assumption, the output has a well defined growth rate. The input growth rate, however, must be some weighted average of the labor growth and capital growth rates; the appropriate weights are shown to be the value shares of labor and capital in national income. The two assumptions are quite restrictive. The use of a single output requires aggregation of commodities and makes it difficult to compare sectors in terms of productivity performance. The notion of perfect competition is a far cry from most observed economies.

I will show how growth accounting can be freed from these assumptions. Basically I will work in a multi-dimensional commodity model and calculate productivities without using observed value shares. The analysis is self contained and serves as a nice refresher of mathematical economics. The main concepts of this branch of economics are equilibrium, efficiency, and the welfare theorems that interrelate equilibrium and efficiency. I will review all this in the next section. To make the theory operational I will then consider the linear case of the model, with constant returns to scale and nonsubstitutability in both production and household consumption. Efficiency is then the outcome of a linear program and the Lagrange multipliers of the factor input constraints measure their productivities. Summing over endowments I obtain total factor productivity. The analysis is shown to be consistent with the aforementioned Solow residual. Moreover, input-output analysis will enable us to reduce total factor productivities growth rates to sectoral productivity and thus to pinpoint the strong and the weak sectors.

4. Equilibrium and efficiency

Denote the number of commodities in an economy by integer n. The commodity space is the n-dimensional Euclidian space, $\mathbb{R}^n$. A commodity bundle is a point in this space, say $\mathbf{y} \in \mathbb{R}^n$. Negative components represent inputs and positive components outputs. For example, in a Robinson Crusoe economy, where (labor) time is transformed into food, $(1, -1)'$ is the bundle representing 1 hour of work and a metric ounce of food. A prime is used to indicate transposition. Denote the collection of all technically feasible commodity bundles by Y. Y is a subset of $\mathbb{R}^n$. It represents the production possibilities of the economy. I make two assumptions on Y. First, Y is convex. This means that if $\mathbf{y}$ and $\mathbf{z}$ belong to Y, then so does $\lambda\mathbf{y} + (1-\lambda)\mathbf{z}$ for any λ between 0 and 1. Although the assumption is always made in general equilibrium analysis, it is not innocent. It rules out increasing returns to scale. Second, Y is compact. In the context of our Euclidian commodity space this means that Y is bounded and closed. In the literature this assumption, namely the boundedness, is relaxed, but at the expense of uninteresting complications.

In a perfectly competitive economy producers pick the production plan that maximizes profit given the prices. Denote the commodity prices by vector $\mathbf{p}$ and let a prime denote transposition. The profit of any production plan $\mathbf{y}$ is then given by $\mathbf{p}'\mathbf{y}$ since inputs have negative signs in $\mathbf{y}$. $\mathbf{p}'\mathbf{y}$ is the inner product of $\mathbf{p}$ and $\mathbf{y}$: $\Sigma_{i=1}^{n} p_i y_i$. Here the positive terms represent revenue and the negative terms cost. Now maximize $\mathbf{p}'\mathbf{y}$ by choosing $\mathbf{y}$. The solution will depend on $\mathbf{p}$ and, therefore, is denoted $\mathbf{y}(\mathbf{p})$. Formally,

$$\mathbf{y}(\mathbf{p}) = \underset{\mathbf{y} \in Y}{\text{argmax}} \ \mathbf{p}'\mathbf{y}$$

Given $\mathbf{p}$, producers "supply" $\mathbf{y}(\mathbf{p})$. Strictly speaking only the positive components represent supply, while the negative components represents business demand, as for labor. I define **supply** as the mapping $\mathbf{y}(\cdot)$. This constitutes one side of equilibrium analysis.

Turn to consumers. For simplicity I assume there is only one utility function, u, so consumers have the same preferences. For a commodity bundle $\mathbf{y}$, the real number $u(\mathbf{y})$ represents the utility it yields to the consumers. Utility is essentially ordinal. Comparing commodity bundles $\mathbf{y}$ and $\mathbf{z}$, what matters is if $u(\mathbf{y}) > u(\mathbf{z})$, $u(\mathbf{y}) < u(\mathbf{z})$, or $u(\mathbf{y}) = u(\mathbf{z})$, but the absolute difference between the utility levels is immaterial. In fact, the entire analysis will be unaffected by a monotonic transformation of the utility function. I make three assumptions on u. First, u is continuous. This is an innocent, technical assumption, that can be shown to be implied by the other assumptions, using a monotonic transformation. The second assumption is that u is increasing. This means that more is preferred. Third, u is quasi-concave. This is defined by the condition that the preferred set, $\{\mathbf{y} | u(\mathbf{y}) \geqslant \text{constant}\}$ is convex. It means that consumers prefer convex combinations.

In a perfectly competitive economy consumers pick the commodity bundle that maximizes utility subject to the budget constraint and given the prices. What is the budget constraint? For a moment, ignore dividends, so that all income stems from labor. In the framework of Robinson Crusoe's economy, the question is when $\mathbf{y} = (-h, f)'$ is financially feasible. (Here h is hours worked and f is amount of food.) If p_2 is the price of good and p_1 the price of labor time, then the answer is $p_2 f \leqslant p_1 h$, which can be written briefly as $\mathbf{p}'\mathbf{y} \leqslant 0$. The budget constraint is basically zero, because the commodity bundle has a negative component that generates (labor) income. In a private enterprise economy, profit, $\mathbf{p}'\mathbf{y}(\mathbf{p})$, supplements the budget constraint and consumers solve the following optimization problem

$$\max_{\mathbf{y}} u(\mathbf{y}) \text{ subject to } \mathbf{p}'\mathbf{y} \leqslant \mathbf{p}'\mathbf{y}(\mathbf{p})$$

The commodity bundle that comes out of this is what consumers "demand." (The positive components represent demand, the negative components household supply, as of labor.) I define **demand** as the mapping from prices $\mathbf{p}$ to the commodity bundle that solves the consumers' problem.

Now we have all the building bricks and can proceed to define the main concepts of mathematical economics, namely equilibrium and efficiency. Conceptually, they are very different. Equilibrium requires a price system; it is defined by the equality between demand and supply. Since the latter are both mappings from prices to commodity bundles, **equilibrium** is defined formally as a price vector, $\mathbf{p}^*$, such that supply and demand assume a common value. Equilibrium is a positive concept, to describe what actually happens in market economies, without saying it is good or bad. Statements on the performance of an economy, however, are normative and require no price mechanism. Suppose we want to compare a centrally planned economy to a decentralized market economy. The centrally planned economy may have no price system at all. Still we want to evaluate which one performs better. This is a matter of utility. We say one economy is better than another if it attains a higher utility level for the consumers. An economy is efficient if it obtains the maximum utility level that is technologically feasible. Since utility is defined on commodity bundles, **efficiency** is defined formally by a commodity bundle, $\mathbf{y}^*$, such that utility is maximized over Y:

$$\mathbf{y}^* = \underset{\mathbf{y}\in Y}{\operatorname{argmax}}\ u(\mathbf{y})$$

Notice the conceptual difference between equilibrium and efficiency. The former is given by a price vector, the latter by a commodity bundle. An equilibrium equates supply and demand, but makes no statement on the level of utility. Efficiency promotes utility, but requires no price system.
Although the concepts are very different, there is a deep, close relationship for perfectly competitive economies. By definition, an economy is perfectly competitive if no producer or consumer can manipulate the prices, but considers them as given. It can be claimed that the commodity bundle generated by the equilibrium price vector is efficient. In short, an equilibrium is efficient. This statement is called the **first welfare theorem**. I also claim that an efficient commodity bundle can be generated by an equilibrium price vector. In short, an efficient allocation is an equilibrium. This statement is called the **second welfare theorem**. The two welfare theorems are deep and must be proved.

The proof of the first welfare theorem is relatively easy. We must show that an equilibrium, say $\mathbf{p}^*$, generates an efficient allocation, $\mathbf{y}(\mathbf{p}^*)$. The proof is by contradiction.

Suppose $\mathbf{y}(\mathbf{p}^*)$ is not efficient. By definition of efficiency there exists $\mathbf{y} \in Y$ such that $u(\mathbf{y}) > u(\mathbf{y}(\mathbf{p}^*))$. By definition of demand it must be that y is too expensive: $\mathbf{p}^{*\prime}\mathbf{y} > \mathbf{p}^{*\prime}\mathbf{y}(\mathbf{p}^*)$. By definition of supply it must be that $\mathbf{y}$ is not feasible: $\mathbf{y} \notin Y$. This contradicts the definition of $\mathbf{y}$. The supposition that $\mathbf{y}(\mathbf{p}^*)$ is not efficient is therefore not tenable. This completes the proof that an equilibrium is efficient.

The proof of the second welfare theorem proceeds as follows. Let $\mathbf{y}^*$ be efficient, hence maximize $u(\mathbf{y})$ over Y. Then we must construct an equilibrium price system that generates it. Consider the feasible set, Y, and the preferred set, $\{\mathbf{y} \in \mathbb{R}^n | u(\mathbf{y}) > u(\mathbf{y}^*)\}$. By efficiency of $\mathbf{y}^*$, the sets do not intersect. By assumptions on production and utility, the two sets are convex. Now we invoke Minkowski's separating hyperplane theorem, by which two convex sets that do not intersect can be separated by a hyperplane. (See, for example, Rockafellar, 1970). Hence there exists a row vector, say $\mathbf{p}^*$, such that

$$\mathbf{p}^{*\prime}\mathbf{y}_1 > \mathbf{p}^{*\prime}\mathbf{y}_2$$

holds for all $\mathbf{y}_1 \in \{\mathbf{y} \in \mathbb{R}^n | u(\mathbf{y}) > u(\mathbf{y}^*)|\}$ and $\mathbf{y}_2 \in Y$. I claim $\mathbf{p}^*$ is an equilibrium. For this we must show that given $\mathbf{p}^*$, $\mathbf{y}^*$ is supplied and demanded. First consider supply. Since utility is increasing, the above inequality yields for any $\boldsymbol{\varepsilon} > \mathbf{0}$ (in $\mathbb{R}^n$)

$$\mathbf{p}^{*\prime}(\mathbf{y}^* + \boldsymbol{\varepsilon}) > \mathbf{p}^{*\prime}\mathbf{y},\ \mathbf{y} \in Y.$$

Hence $\mathbf{p}^{*\prime}\mathbf{y}^* \geqslant \mathbf{p}^{*\prime}\mathbf{y}$, hence $\mathbf{y}^*$ maximizes profit and, therefore, is supplied: $\mathbf{y}^* = \mathbf{y}(\mathbf{p}^*)$. Next consider demand. If $\mathbf{y}$ is superior to $\mathbf{y}^*$, $u(\mathbf{y}) > u(\mathbf{y}^*)$, then it is out of the budget, $\mathbf{p}^{*\prime}\mathbf{y} > \mathbf{p}^{*\prime}\mathbf{y}^* = \mathbf{p}^{*\prime}\mathbf{y}(\mathbf{p}^*)$. Hence $\mathbf{y}^*$ maximizes utility subject to the budget constraint and, therefore, is demanded. This completes the proof that an efficient allocation is an equilibrium.

So far, I have remained silent about existence. Does an equilibrium exist? The usual analysis to find an intersection point of supply and demand is by means of a so called fixed point theorem. This is difficult. We make a shortcut. It is easy to see that an efficient allocation exists. All we have to do is to maximize utility, u over the feasible set, Y. Since u, is continuous and Y is compact, a maximum exists, say $\mathbf{y}^*$. By the second welfare theorem it is an equilibrium, say $\mathbf{p}^*$. Hence an equilibrium exists.

In the literature all sorts of variations on the above analysis are found. More commodities, more products, more consumers, you name it. The basic structure, however, remains the same. Equilibrium is defined by the equality of supply and demand, efficiency by the impossibility to raise the utility level further, and the two are related by the first and second welfare theorem provided convexity assumptions hold and agents are price takers. Then competitive prices can be analyzed by studying the efficiency problem, where utility is maximized over the feasible set. For example, the well-known statement that competitive economies reward factor inputs according to their productivities can be demonstrated. This will be done in the next section for linear economies.

5. Efficiency and productivity

The model of the last section is quite general, at least in terms of functional forms. I now add the flesh and blood of linear economics, including input-output analysis. Let there be m activities. Denote an $m \times n$-dimensional matrix of outputs by $\mathbf{V}$ and an $n \times m$-dimensional matrix of inputs by $\mathbf{U}$. Add an m-dimensional vector of capital inputs, $\mathbf{k} \geqslant \mathbf{0}$, and similarly for labor, $\mathbf{l} \geqslant \mathbf{0}$. Assume every activity requires positive factor input (k_i and l_i not both zero). Let the economy be endowed with a capital stock k and labor force l. Let

$$Y = \{\mathbf{y} \in \mathbb{R}^n | \mathbf{y} \leqslant (\mathbf{V}' - \mathbf{U})\mathbf{s},\ \mathbf{k}'\mathbf{s} \leq k,\ \mathbf{l}'\mathbf{s} \leqslant l, \mathbf{s} \geqslant \mathbf{0}\}$$

where $\mathbf{s} \in \mathbb{R}^m$ is the vector listing m activity levels. Then Y is an example of a production possibility set as we defined it in Section 3. Y is the intersection of a number of half-spaces, which is obviously convex. The assumption that every activity requires factor input ensures that Y is compact.

The modelling of household consumption is similar. Denote an n-dimensional vector of consumption coefficients by $\mathbf{a} > \mathbf{0}$. Then for $\mathbf{y} \geqslant \mathbf{0}$,

$$u(\mathbf{y}) = \min y_i / a_i$$

is the Leontief utility function. (I choose this utility function, because it enables us to substitute observed consumption values in the Total Factor Productivity growth expression of the next section.) Basically consumers want their bundle in the proportions of $\mathbf{a}$, say $c\mathbf{a}$, where c is a scalar. It is easy to see that

$$u(\mathbf{y}) = \max_{c\mathbf{a} \leqslant \mathbf{y}} c$$

Proof. First we prove $u(\mathbf{y}) \geqslant \max_{c\mathbf{a} \leqslant \mathbf{y}} c$. For all $\mathbf{y} \geqslant c\mathbf{a}$, $u(\mathbf{y}) \geqslant u(c\mathbf{a}) = c$. Hence also $u(\mathbf{y}) \geqslant \max_{c\mathbf{a} \leqslant \mathbf{y}} c$. Next we prove the converse. At least one constraint in $\max_{c\mathbf{a} \leqslant \mathbf{y}} c$ is binding: $a_j c^* = y_j$ for some j, where c^* is the constrained maximum. Now $u(\mathbf{y}) = \min y_i / a_i \leqslant y_j / a_j = c^* = \max_{c\mathbf{a} \leqslant \mathbf{y}} c$. This completes the proof.

We have production and utility, so we can set up the efficiency problem,

$$\max_{\mathbf{y}\in Y} u(\mathbf{y})$$

Using the alternative formulation of the utility function, we can rewrite the efficiency problem as

$$\max_{\mathbf{s},\mathbf{y},c} c \text{ subject to } c\mathbf{a} \leqslant \mathbf{y} \text{ and } \mathbf{y} \in Y.$$

Notice that both the objective and the constraints are linear in the variables. The efficiency problem of a linear economy is a linear program. The linear program can be simplified slightly by eliminating one of the variables, $\mathbf{y}$:

$$\max_{\mathbf{s},c} c \text{ subject to } c\mathbf{a} \leqslant (\mathbf{V}' - \mathbf{U})\mathbf{s},\ \mathbf{k}'\mathbf{s} \leqslant k,\ \mathbf{l}'\mathbf{s} \leqslant l,\ \mathbf{s} \geqslant \mathbf{0}$$

This linear program maximizes the level of final consumption subject to the material balance, the capital and labor constraints, and a nonnegativity constraint. Another, succinct formulation of the linear program, is

$$\max(\mathbf{0}'\ 1)\begin{pmatrix}\mathbf{s}\\ c\end{pmatrix} \text{ subject to } \begin{bmatrix}\mathbf{U}-\mathbf{V}' & \mathbf{a}\\ \mathbf{k}' & 0\\ \mathbf{l}' & 0\\ -\mathbf{I} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{s}\\ c\end{bmatrix} \leqslant \begin{bmatrix}\mathbf{0}\\ k\\ l\\ \mathbf{0}\end{bmatrix}$$

In general, when we max $f(\mathbf{x})$ subject to $g(\mathbf{x}) \leqslant b$, the first order conditions are $\mathbf{f}' = \lambda\mathbf{g}'$, $\lambda \geqslant 0$. Here $\mathbf{f}'$ is the (row) vector of partial derivatives $\partial f/\partial x_i$ of f. If g is scalar valued, $\mathbf{g}'$ is also the row vector of partial derivatives $\partial g/\partial x_i$. If the constraints are given by $G(\mathbf{x}) \leqslant \mathbf{b}$, with G vector valued, the first order conditions are $\mathbf{f}' = \boldsymbol{\lambda}'\mathbf{G}$, $\boldsymbol{\lambda} \geqslant \mathbf{0}$, where $\mathbf{G}$ is the Jacobian matrix of partial derivatives (i.e. element g_{ij} of matrix $\mathbf{G}$ equals $\partial g_i(\mathbf{x})/\partial x_j$.)

[See Word doc for graph.]

The first order conditions reflect the tangency of the isoquants of the objective and constraint functions. In the picture f and g grow in the same direction (the North-East), hence $\lambda \geqslant 0$. If λ were negative, then f and g would grow in opposite directions and one could simply increase f by wandering into the feasible region (g would be reduced). λ is called the Lagrange multiplier. Because $\mathbf{f}' = \lambda\mathbf{g}'$, and $g(\mathbf{x}) \leqslant b$, λ measures the rate of change of the objective function with respect to the constraint. If b is relaxed by one unit, then f goes up by λ units. If G is vector valued, then each constraint has a Lagrange multiplier and $\boldsymbol{\lambda}$ is a vector of Lagrange multipliers.

In our linear program, $f\begin{pmatrix}\mathbf{s}\\ c\end{pmatrix} = (\mathbf{0}'\ 1)\begin{pmatrix}\mathbf{s}\\ c\end{pmatrix}$ and $\mathbf{f}' = (\mathbf{0}'\ 1)$. Also,

$$G\begin{pmatrix}\mathbf{s}\\ c\end{pmatrix} = \begin{bmatrix} \mathbf{U}-\mathbf{V}' & \mathbf{a} \\ \mathbf{k}' & 0 \\ \mathbf{l}' & 0 \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}\begin{pmatrix}\mathbf{s}\\ c\end{pmatrix} \text{ and } \mathbf{G} = \begin{bmatrix} \mathbf{U}-\mathbf{V}' & \mathbf{a} \\ \mathbf{k}' & 0 \\ \mathbf{l}' & 0 \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}.$$

The constraints are the material balance, the capital constraint, the labor constraint, and the nonnegativity constraint. It is customary to denote the Lagrange multipliers by $\mathbf{p}$, r, w, and $\boldsymbol{\sigma}$, respectively. The first order conditions, $\mathbf{f}' = \boldsymbol{\lambda}'\mathbf{G}$, $\boldsymbol{\lambda} \geqslant \mathbf{0}$ read

$$(\mathbf{0}'\ 1) = (\mathbf{p}',\ r,\ w,\ \boldsymbol{\sigma}')\begin{bmatrix} \mathbf{U}-\mathbf{V}' & \mathbf{a} \\ \mathbf{k}' & 0 \\ \mathbf{l}' & 0 \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}, \quad (\mathbf{p}',\ r,\ w,\ \boldsymbol{\sigma}') \geqslant \mathbf{0}'$$

The second component, $\mathbf{p}'\mathbf{a} = 1$, is a price normalization condition. The first component, $\mathbf{0}' = \mathbf{p}'(\mathbf{U}-\mathbf{V}') + r\mathbf{k}' + w\mathbf{l}' - \boldsymbol{\sigma}'$, can be rewritten as $\mathbf{p}'(\mathbf{V}'-\mathbf{U}) = r\mathbf{k}' + w\mathbf{l}' - \boldsymbol{\sigma}$, $\boldsymbol{\sigma} \geqslant \mathbf{0}$, or

$$\mathbf{p}'(\mathbf{V}'-\mathbf{U}) \leqslant r\mathbf{k}' + w\mathbf{l}'$$

On the left hand side we find value-added and on the right hand side factor costs, for the respective activities.

$\mathbf{p}$, r and w are the perfectly competitive equilibrium prices. I am going to demonstrate this by means of the so called phenomenon of complementary slackness. Let me explain this phenomenon in terms of $\max f(\mathbf{x})$ subject to $G(\mathbf{x}) \leqslant \mathbf{b}$. The first order conditions are $\mathbf{f}' = \boldsymbol{\lambda}'\mathbf{G}$, $\boldsymbol{\lambda} \geqslant \mathbf{0}$. The phenomenon says that if a constraint is non-binding, $g_i(\mathbf{x}) < b_i$, then the Lagrange multiplier is zero, $\lambda_i = 0$. Hence g_i plays no role in the first order condition. The phenomenon also says, that if a Lagrange multiplier is strictly positive, $\lambda_i > 0$, then the constraint is binding, $g_i(\mathbf{x}) = b_i$. A nice way to write the phenomenon of complementary slackness is

$$\boldsymbol{\lambda}'[G(\mathbf{x}) - \mathbf{b}] = 0.$$

The left hand side is the inner product of two nonnegative vectors. It is zero if and only if each term of the inner product is zero: $\lambda_i[g_i(\mathbf{x}) - b_i] = 0$. This, indeed, is a short way of writing $g_i(\mathbf{x}) < b_i \Rightarrow \lambda_i = 0$ and $\lambda_i > 0 \Rightarrow g_i(\mathbf{x}) = b_i$.

Now I explain why the Lagrange multipliers are competitive prices. Suppose that for some activity value-added is strictly less than factor costs. Then $\sigma_i > 0$. By the phenomenon of complementary slackness, $s_i = 0$. Hence the price system is such that negative profits signal activities that are inactive in the coefficient allocation. If the economy would have this price system and producers are profit maximizers, they would undertake precisely those activities which we want them to do. Notice that profits would be zero: The unprofitable activities are inactive, and value-added is everywhere less than or equal to factor costs.

There is another interesting consequence of the phenomenon of complementary slackness, namely the identity between national product and national income. If G is linear, $G(\mathbf{x}) = \mathbf{G}\mathbf{x}$ and the last equation becomes

$$\boldsymbol{\lambda}'\mathbf{G}\mathbf{x} = \boldsymbol{\lambda}\mathbf{b}$$

By the first order condition, $\mathbf{f}' = \boldsymbol{\lambda}'\mathbf{G}$,

$$\mathbf{f}'\mathbf{x} = \boldsymbol{\lambda}\mathbf{b}$$

If f is also linear, this reads

$$f(\mathbf{x}) = \boldsymbol{\lambda}\mathbf{b}$$

In our linear program,

$$(\mathbf{0}'\ 1)\begin{pmatrix}\mathbf{s}\\ c\end{pmatrix} = (\mathbf{p}',\ r,\ w,\ \boldsymbol{\sigma}')\begin{bmatrix}\mathbf{0}\\ k\\ l\\ \mathbf{0}\end{bmatrix}$$

or

$$c = rk + wl$$

This is the famous macro-economic identity of the national product and national income. It confirms that Lagrange multipliers measure the rate of change of the objective function (consumption level c) with respect to the constraints (capital k and labor l). If the stock of capital is increased by a unit, then the contribution to the objective is r. Hence r measures the productivity of capital. Similarly w measures the productivity of labor. r and w need not be the observed prices of capital and labor, but are the Lagrange multipliers of the efficiency program, also called shadow prices. For perfectly competitive economies, however, there is agreement.

6. Total factor productivity

Capital productivity is r and labor productivity is w where r and w are the shadow prices of the linear program that maximizes consumption subject to the material balance, the capital constraint, the labor constraint, and the nonnegativity constraint. Now let time evolve. Everything changes, not only the output levels, but also the technical coefficients and the consumption coefficients. The linear program changes. r and w change. Hence there is capital productivity growth, $\dot{r} = dr/dt$, and labor productivity growth, $\dot{w} = dw/dt$. All this is per unit of capital or labor. Total capital productivity growth is $\dot{r}k$ and total labor productivity growth is $\dot{w}l$. Normalizing by the level, we obtain the nominal total factor productivity growth rate, $(\dot{r}k+\dot{w}l)/(rk+wl)$. To obtain it in real terms we must subtract the price increase of the consumption bundle, $\dot{p}a$. The (real) **total factor productivity growth rate** is

$$TFP = (\dot{r}k + \dot{w}l)/(rk + wl) - \dot{p}a.$$

Here k and l are the factor constraints and r and w their Lagrange multipliers; a is the vector of consumption coefficients. Although this productivity growth concept is grounded in the theory of mathematical programming (where Lagrange multipliers measure productivities of constraints), there is perfect consistency with the traditional Solow residual. Recall the macro-economic identity of the national product and national income, $c = rk + wl$. divandiding though by the identity itself, we obtain Total differentiation yields $\dot{r}k + \dot{w}l = \dot{c} - r\dot{k} - w\dot{l}$ and division by the identity itself leads to

$$TFP = \dot{c}/c - \dot{p}a - r\dot{k}/(rk + wl) - w\dot{l}/(rk + wl)$$

If we use shorthand $\hat{c} = \dot{c}/c$ for a relative growth rate, we obtain

$$TFP = \hat{c} - \dot{p}a - \alpha_k\hat{k} - \alpha_l\hat{l}$$

where $\alpha_k = rk/(rk+wl)$, the competitive value share of capital, and $\alpha_l = wl/(rk+wl)$, the competitive value share of labor. The right hand side of the last equation is precisely the Solow residual. Notice, however, that the competitive value shares are not necessarily the observed ones. For noncompetitive economies, they must be calculated by means of the linear program of Section 4; for an application see ten Raa and Mohnen (2000).

7. Input-output analysis of total factor productivity

By definition, positive TFP-growth means that output grows at a faster rate than input and, therefore, that the output/input ratio or standard of living goes up. In this section I will explain the phenomenon in terms of technical change at the sectoral level.

The linear program selects activities to produce the required net output of the economy. In continuous time we may consider infinitesimal changes and the pattern of activities that are actually used is locally constant (except in degenerate points where the linear program has multiple solutions). In this section we ignore the activities that are not used. Hence, activity vector $\mathbf{s}$ is and remains positive.

From the last section, the Solow residual is

$$TFP = \dot{c}/c - \dot{p}a - r\dot{k}/(rk + wl) - w\dot{l}/(rk + wl) = (\dot{c} - \dot{p}ac - r\dot{k} - w\dot{l})/c$$

. We are going to express c, k and l in terms of $\mathbf{s}$. By complementary slackness between $c\mathbf{a} \leqslant (\mathbf{V}' - \mathbf{U})\mathbf{s}$ and $\mathbf{p} \geqslant \mathbf{0}$ we have $c\mathbf{p}'\mathbf{a} = \mathbf{p}'(\mathbf{V}' - \mathbf{U})\mathbf{s}$, or, using the price normalization condition,

$$c = \mathbf{p}'(\mathbf{V}' - \mathbf{U})\mathbf{s}$$

Assume that capital and labor have positive productivity. Then, also by complementary slackness,

$$k = \mathbf{k}'\mathbf{s},\ l = \mathbf{l}'\mathbf{s}$$

Substitution yields

$$TFP = \{[\mathbf{p}'(\mathbf{V}' - \mathbf{U})]^{\cdot} - r\dot{\mathbf{k}}' - w\dot{\mathbf{l}}'\}\mathbf{s}/c - \dot{p}a + [\mathbf{p}'(\mathbf{V}' - \mathbf{U}) - r\mathbf{k}' - w\mathbf{l}']\dot{\mathbf{s}}/c$$

Since $\mathbf{s}$ remains positive, by complementary slackness, $\mathbf{p}'(\mathbf{V}' - \mathbf{U}) = r\mathbf{k}' - w\mathbf{l}'$, and the second term vanishes. It is customary to define TFP-growth of sector i as output growth minus input growth, normalized by output:

$$TFP_i = \frac{[\mathbf{p}'(\mathbf{V}' - \mathbf{U})^{\cdot}]_i - r\dot{k}_i - w\dot{l}_i}{(\mathbf{p}'\mathbf{V}')_i}$$

It then follows that

$$\begin{aligned} TFP &= \sum_i (\mathbf{p}'\mathbf{V}')_i TFP_i s_i / c + [\dot{\mathbf{p}}'(\mathbf{V}' - \mathbf{U})\mathbf{s} - \dot{p}ac]/c \\ &= \sum_i d_i TFP_i \end{aligned}$$

where $d_i = (\mathbf{p}'\mathbf{V}')_i s_i / \mathbf{p}'(\mathbf{V}' - \mathbf{U})\mathbf{s}$ and the remainder vanishes because of the material balance, assumed to be binding. These weights d_i are called Domar weights and sum to the gross/net output ratio of the economy, which is greater then one.

To see the reduction of TFP-growth as reductions of input-output coefficients in the traditional sense, consider the case where sectors produce single outputs. Then

$$TFP_i = (p_i \dot{v}_{ij(i)} - \sum_j p_j \dot{u}_{ji} - r\dot{k}_i - w\dot{l}_i)/(p_i v_{j(i)})$$

In this case input coefficients are defined by $a_{ji} = u_{ji}/v_{ij(i)}$, $\kappa_i = k_i/v_{ij(i)}$ and $\mu_i = l_i/v_{ij(i)}$. Substitution yields

$$TFP_i = [p_i \dot{v}_{ij(i)} - \sum_j p_j (a_{ji} v_{ij(i)})^{\cdot} - r(\kappa_i v_{ij(i)})^{\cdot} - w(\mu_i v_{ij(i)})^{\cdot}]/(p_{j(i)} v_{ij(i)})$$

By complementary slackness, $p_i v_{ij(i)} = \sum_j p_j a_{ji} v_{ij(i)} - r\kappa_i v_{ij(i)} - w\mu_i v_{ij(i)}$ and we obtain

$$TFP_i = (-\sum_j p_j \dot{a}_{ji} - r\dot{\kappa}_i - w\dot{\mu}_i)/p_{j(i)},$$

that is sectoral cost reductions. With obvious matrix notation,

$$TFP = -\sum_i d_i (\mathbf{p}'\dot{\mathbf{a}}_i - r\dot{\boldsymbol{\kappa}} - w\dot{\boldsymbol{\mu}})/p_{j(i)}$$

is reduced to reduction in input-output coefficients. If there is only one sector producing each commodity, then $j(i) = i$. If sectors produce multiple outputs, then the result basically holds, but input-output coefficients are no longer obtained by simple scalar divisions.

8. Conclusion

For perfectly competitive economies there is an intimate relationship between efficiency and equilibrium. The marginal productivities of capital and labor that are the Lagrange multipliers to the efficiency program coincide with perfectly competitive equilibrium prices. For such economies one can measure TFP-growth by means of the Solow residual, using the observed value share of the factor inputs. Most economies, however, are not perfectly competitive. Then, to measure productivities, one must find the shadow prices of the factor inputs by solving a linear program. In this paper I proposed the linear program that maximizes Leontief utility subject to resource constraints. We thus obtained a Solow residual measure for TFP without assuming that the economy is on its frontier. The flipside of the coin is that the numerical values we use in the residual reflect shadow prices instead of observed prices. The data required for the determination of TFP are input-output coefficients and constraints on capital and labor. These data capture the structure of the economy and are real rather than nominal. Our measure of TFP-growth, firmly grounded in the theory of mathematical programming, admits a decomposition in sectoral contributions, allowing us to pinpoint the strong and the weak sectors of the economy.

We have freed neoclassical growth accounting from its use of market values of factor inputs in the evaluation of the Solow residual and, therefore, some circularity in its methodology. Perhaps surprisingly, we accomplished this by using input-output analysis to determine the values of factor inputs. Input-output analysis and neoclassical economics can be used fruitfully to fill gaps in each other. Contrary to perception, the gap in input-output analysis is not the interaction between prices and quantities, but the concept of marginal productivity, and the gap in neoclassical economics is not the structure of the economy, but the determination of value shares of factor inputs.

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