Increasing returns and perfect competition: the role of land

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Abstract

The classical inconsistency between increasing returns and perfect competition is studied. For example, if firms must pay a fixed cost of entry but then can produce using a constant returns to scale technology, they will generally operate at a loss, necessitating a government subsidy in order to attain an efficient allocation. Here we provide examples demonstrating that perfect competition and increasing returns can be consistent by extending the Alonso model to include production. The key is that producers use intervals of land, and the price they pay for land interior to the parcels can be adjusted to provide an implicit subsidy.
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1. Introduction

Our goal is to begin to reconcile the notions of increasing returns and perfect competition. We demonstrate in our model that equilibria can exist and can be efficient without government intervention. This finding is established for a rather specific model with parameter restrictions. Land plays a key role in our analysis. In this context, models of imperfect competition have been analyzed and are known to produce market failures. It

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is not known, however, if such failures are due to product differentiation or to the departure from price taking behavior. We address this issue by assuming that agents take prices as given.

It is well known that global increasing returns (say, a fixed cost followed by constant returns to scale production) and perfect competition are not compatible, since at an equilibrium, the first-order condition for profit maximization—price equals marginal cost—implies negative profits. Although substantial progress has been made using models in which price is set at marginal cost but firms are subsidized, or multipart tariffs are employed, problems still remain; see Bonnisseau and Cornet [14] (as well as other papers in the symposium issue), Bonanno [13] or Vassilakis [45,46] for discussion.[1]

Our initial goal was to prove a second welfare theorem. Here transfers have generally been employed in the literature. They can obviously mitigate the problem of negative profits for producers by simply providing a subsidy to producers who are operating at a Pareto optimum but who would otherwise make a loss at supporting prices. The idea that firms yielding increasing returns to scale should be subsidized in order to obtain an efficient allocation goes back at least to Marshall [29, Book V, Chapter XIII], the first edition of which was published in 1890. A precursor can be found in Whitaker [47, pp. 88–89, 228–230], who published writings of Marshall dating from the 1870s. Pigou [35, Part II, Chapter XI], first published in 1920, touches on this subject in passing. Pigou [33, p. 197] is particularly explicit:[2]

> In order to maximize satisfaction—inequalities of wealth among different people and so on being ignored—it is necessary, except in the special case where satisfaction is maximised by a nil output, for that quantity of output to be produced which makes demand price equal to marginal costs, *i.e.* which corresponds to the point of intersection of the demand curve and the curve of marginal costs. [. . .] Output, however, *tends* to be carried to the point in respect of which the demand curve intersects with the supply curve. [. . .] But in conditions of decreasing costs, where the supply curve coincides with the curve of average costs, it will not be the right point. Unless the State intervenes by a bounty or in some other way, output will be carried *less far* than it is socially desirable that it should be carried.

It is important to note that the work of Marshall and Pigou confused scale economies with externalities internal to an industry but external to each firm, and consequently they recommended a misplaced Pigouvian remedy for scale economies. Our reconciliation of increasing returns and perfect competition is direct and invokes no externality argument.

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[1] For instance, marginal cost pricing relates only to the first-order conditions for optimization for the firms, so at a marginal cost pricing equilibrium, a firm may not be maximizing profits. Further, a marginal cost pricing or multipart tariff equilibrium allocation is not necessarily Pareto optimal. (Marginal cost pricing reflects the first-order conditions for Pareto efficiency, but the second-order conditions might not hold.)

[2] Pigou [33] is part of a far-ranging discussion about "Empty Boxes" in the *Economic Journal* addressing this topic; see, in particular, Robertson [37, p. 22]. Others involved in this discussion are Clapham [15], Pigou [32,34], Sraffa [42,43], Shove [40,41], Robbins [36], Schumpeter [39], Young [48], and Robertson [38].

The use of transfers would be an easy way out of the conflict between increasing returns and a perfectly competitive equilibrium by essentially assuming the conflict away. Instead, we focus on existence of a competitive equilibrium and the first welfare theorem.

This research has applications to the theory of agglomeration and city formation. Increasing returns is often used as an agglomerative force in models seeking to explain how, where, and why cities form. For example, Fujita [19]; Fujita and Krugman [21,22], and Krugman [26–28], which were preceded by Abdel-Rahman [1,2] and Abdel-Rahman and Fujita [3], use a Dixit–Stiglitz [17] framework and increasing returns to generate city formation in a monopolistic competition context. Since our model will employ increasing returns in a spatial context, it offers the prospect of addressing questions and generating testable hypotheses about cities. This is discussed further in the conclusion.

In what follows, we stick as closely as possible to the perfectly competitive ideal, since it is simplest to analyze, it is a very standard and convenient benchmark, it allows us to develop proofs of existence of equilibrium (perhaps useful in the imperfect competition context) without having to worry about other distractions, it may be a good approximation to reality in large economies, and it will tell us when the welfare theorems are likely to hold and why. Moreover, it enables us to separate problems due to the spatial context from those attributable to imperfect competition. Notice that models of marginal cost pricing, multipart tariffs, and subsidization of firms under increasing returns all employ close relatives of perfect competition.

We investigate whether a government ought to intervene in markets for commodities subject to increasing returns in production. The key to the analysis is provided by Berliant and Fujita [9], who show that for Alonso's urban economic model, a model of pure exchange on the real line where agents are required to own intervals that represent land parcels, there is generally a continuum of equilibria under perfect competition.[3] Infra-marginal land (that is, land not at the endpoints of an interval owned by an agent) is not priced uniquely, thus allowing a kind of indeterminacy in the expenditure of agents on land. It is this kind of indeterminacy that we exploit below to effect *implicit* transfers to producers (by keeping the infra-marginal price of land low) who would otherwise have negative profits.

Section 2 presents the notation and model while Section 3 introduces an example with one producer and one consumer, solving for two different types of equilibria. Section 4 shows how these equilibria can be extended to a model with two producers and multiple

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[3] A spatial model with finite numbers of producers and consumers (rather than a continuum) is examined because in the arguments we use, agents employ intervals rather than densities of land. By this, we mean that agents own land parcels represented by sets of positive Lebesgue measure in a Euclidean space (**R**) rather than owning parcels represented by a quantity at a point. The latter is more common in urban economics, and is usually called a density. Berliant [7] shows that the usual approximation of continuum economies by finite economies does not work when land plays a role in the models, so demand and equilibria of the continuum models may not be close to those of any interesting finite model. It is then reasonable to ask if the continuum models make any sense. Examples in Berliant and ten Raa [11] show that equilibrium can fail to exist in the monocentric city model under standard assumptions on preferences. Examples in Berliant et al. [10] show that the welfare theorems can fail in the monocentric city model. Berliant and Wang [12] show that even utilitarian social optima might fail to exist in continuum models with land. The implication of these examples is that the use of a continuum of consumers solves some of the problems associated with the indivisibility of location, but creates others.

consumers, Section 5 presents a version of the first welfare theorem, while Section 6 concludes. Appendix A contains all of the proofs.

2. The model

We introduce production into Alonso's [4] model of pure exchange. The model of pure exchange was developed further by Asami [5], Asami et al. [6], Berliant [8], and Berliant and Fujita [9].

Consider a long narrow city represented on the real line. Land is given by $X = [0, l)$, where l is the length of the city. In Section 4, it will be convenient to use another interval of the real line for X to reduce computations. The density of land available is 1 at each point $x \in X$.

There are $i = 1, \ldots, I$ consumers and $j = 1, \ldots, J$ producers. Each consumer has an endowment of 1 unit of labor, which will be supplied inelastically. For simplicity, labor is assumed to be homogeneous, so labor income is the same for all consumers. Moreover, consumers all have the same preferences, and will get utility from a composite consumption good and land. Thus, $u : \mathbf{R}_+^2 \to \mathbf{R}$. Consumers are not endowed with composite good or land. Composite good is produced, while an absentee landlord is endowed with land. We write $u(c, s)$, where c is the quantity of consumption good and s is the quantity of land consumed; the latter is equal to the length of the interval owned by the consumer. For consumer i, c_i is composite good consumption, s_i is land consumption, w is the wage rate, and $[a_i, a_i + s_i) \subseteq X$ is the parcel of land owned by i.

Notice that w is assumed to be independent of the location of labor. This is an assumption of perfect competition, that each agent takes prices as given independent of their own actions and the actions of other agents, particularly firms' locations. Without such an assumption, equilibrium allocations are not likely to be Pareto optimal. Since our purpose is to reconcile increasing returns with perfect competition, we must take prices as parametric. Of course, for other purposes, imperfect competition is a more suitable premise. If wages are allowed to vary with location in the context of perfect competition, then the constant wage gradient equilibrium that we study here naturally becomes a special case.[4] Consumers have no intrinsic preference for location.

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[4] The decision whether or not to use a wage gradient is not at all obvious. Our model is not one of multiple regions, but rather of one city, since we have commuting cost but no transport cost. From a positive viewpoint, one does not observe in the real world wages paid to workers differing by their location of residence within a city or by producer location within a city. From a normative viewpoint, if we had wages differing by producer, our equilibrium allocations would likely not be efficient, since symmetry of the allocation would be destroyed. In the literature, for example, Fujita and Ogawa [23] use a wage gradient that differs by location of a firm (but not by location of consumer residence). Subject to the remarks above, such a structure would be admissible in our framework, but would make the analysis much messier. In general, addition of a wage gradient to a model will not add extra degrees of freedom to equilibrium determination. Although more free variables are added to the system in the form of wages depending on locations, extra market clearing conditions equating labor demand to supply at each location are also added.

Composite consumption good, assumed to be freely mobile, is taken to be numeraire. The price of land is denoted by an integrable function $p: X \to \mathbf{R}$. The price of consumer i's parcel is $\int_{a_i}^{a_i+s_i} p(x) \, dm(x)$. Throughout, m is the Lebesgue measure on the real line.

Since the labor market is competitive and consumers pay their own commuting cost, consumers will turn to the producers who minimize their commuting cost. Let producer j use land parcel $[b_j, b_j + \sigma_j) \subseteq X$. Define $t > 0$ to be the constant marginal monetary cost (in terms of composite consumption good) of commuting an extra mile. Then the cost of commuting to producer j is given by

$$T_i^j(a_i, s_i, b_j, \sigma_j) = t \cdot \inf\{\|x - y\| \mid x \in [a_i, a_i + s_i), \ y \in [b_j, b_j + \sigma_j)\},$$

the closest point distance between consumer i and employer j. When consumers optimize utility subject to their budget constraints, they will choose to commute to the closest producer. However, we must account for the possibility that there is more than one closest producer.

This is the form of commuting cost used by Alonso [4] and Berliant and Fujita [9]; it incorporates a constant marginal cost of transport per unit distance to the closest firm. Notice that commuting cost depends on both the consumer location and the location of the nearest employer.

The minimal commuting cost available to consumer i is given by

$$\min_j T_i^j(a_i, s_i, b_j, \sigma_j).$$

For notational simplicity, define $B = [b_1, \sigma_1, b_2, \sigma_2, \ldots, b_J, \sigma_J]$ and

$$T_i(a_i, s_i, B) = \left[T_i^1(a_i, s_i, b_1, \sigma_1), \ldots, T_i^J(a_i, s_i, b_J, \sigma_J) \right].$$

The fact that T_i^j can depend on the allocation of land to producer j creates an externality, in that the choice of land parcel by an agent (in particular, a producer) can affect the budget constraint of another (in particular, a consumer). What is fascinating about this observation is that, as we shall see in Section 5, this externality might not create a market failure.

Let Q_i be a J-dimensional unit vector (one component 1 and all others 0), to indicate consumer i's choice of employer. Let S be the collection of all such unit vectors, and let Q_i^j denote component j of Q_i.

Consumer i's optimization problem is[5]

$$\max_{a_i, s_i, c_i, Q_i} u(c_i, s_i)$$

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[5] Unlike most of the literature in urban economics, we do not introduce or use the concept of "bid rent," since we have no need for it. The results and proofs are more easily given in primal rather than dual form. Any references to "marginal willingness to pay" for land are simply to the marginal rates of substitution at a particular bundle of commodities. Notice that agents take into account the total supply of land when solving their optimization problems. This constriction of the commodity space is essential to our results, and appears in the spatial economic literature more generally. It is hard to imagine that a consumer visualizes simultaneously purchasing two different houses on the same parcel or buying a house in a lake when solving her optimization problem.

$$\text{s.t.} \quad c_i + \int\limits_{a_i}^{a_i+s_i} p(x)\,\mathrm{d}m(x) + Q_i \cdot T_i(a_i, s_i, B) = w. \tag{1}$$

This framework allows consumers to choose to work at a firm so that commuting cost and commuting distance are minimized.[6]

Producers use land and labor to produce composite good. All producers have the same production function $g : \mathbf{R}_+^2 \to \mathbf{R}$. Let producer j use land parcel $[b_j, b_j + \sigma_j) \subseteq X$. The scalar $q_j \in \mathbf{R}_+$ represents the labor demand of firm j. We define output of firm j to be $z_j = g(\sigma_j, q_j)$. We assume throughout most of the sequel that $g(\sigma, q) = \beta \cdot \min(\sigma, q) - f$ for $\sigma > 0$ and $q > 0$, where f is a fixed cost in terms of composite good. We define $g(0, 0) = 0$, so it is possible for a firm to shut down. This has the implication that in equilibrium, profits must be non-negative. The only part of this paper where we alter this production function is at the beginning of Section 6, where it is convenient to normalize the labor input for computational purposes. The profit optimization problem of firm j is

$$\pi_j = \max_{b_j, \sigma_j, q_j} g(\sigma_j, q_j) - \int\limits_{b_j}^{b_j+\sigma_j} p(x)\,\mathrm{d}m(x) - q_j w. \tag{2}$$

List the firms' profits in the vector $\pi \equiv [\pi_1, \ldots, \pi_J]$.

We have assumed, implicitly, that only the size of an interval matters in production. Thus, output is a function of land and labor where both inputs are represented by scalars and, therefore, returns to scale can be defined as usual. It is the fixed cost f that gives us increasing returns to scale. The particular form of the production function that we use implies that average cost is globally decreasing, so increasing returns are in fact global.

Following Alonso [4] and the new urban economics literature, an absentee landlord is endowed with all of the land, but gets utility only from composite good. For simplicity, we also endow the absentee landlord with all of the shares in all of the firms.[7] In equilibrium, the absentee landlord collects all of the land rent. Taking $p(\cdot)$ and π as given, the landlord consumes $\int_0^l p(x)\,\mathrm{d}m(x) + \sum_{j=1}^J \pi_j$. The composite good consumption of the landlord will be denoted by c_L.

Notice that, as in the Alonso model, preferences and production are location independent.

We continue with the analogs of standard definitions for this model.

Definition 1. An *allocation* is a list $[(c_i, a_i, s_i, Q_i)_{i=1}^I, c_L, (z_j, b_j, \sigma_j, q_j)_{j=1}^J]$, where for every $i = 1, \ldots, I$ and $j = 1, \ldots, J$, $c_i, z_j, c_L, q_j \in \mathbf{R}_+$, $s_i, a_i, b_j, \sigma_j \in X$, and $Q_i \in S$.

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[6] Strictly speaking, a consumer could choose not to work, but then good consumption would be zero and utility would be suboptimal in all theorems of this paper. Hence we ignore the possibility $Q_i = 0$. Also notice that utility levels will be equal across consumers in equilibrium.

[7] It seems clear that one could allow consumer ownership of stock in the firms without altering the results much, but at the cost of complicating the arguments and notation.

Definition 2. An allocation $[(c_i, a_i, s_i, Q_i)_{i=1}^I, c_L, (z_j, b_j, \sigma_j, q_j)_{j=1}^J]$ is called *feasible* if[8]

$$\sum_{i=1}^I \big[c_i + Q_i \cdot T_i(a_i, s_i, B)\big] + c_L \leqslant \sum_{j=1}^J z_j, \tag{3}$$

$$z_j = g(\sigma_j, q_j) \quad \text{for } j = 1, \dots, J, \tag{4}$$

$$\sum_{i=1}^I Q_i^j = q_j \quad \text{for } j = 1, \dots, J, \tag{5}$$

$$\big([a_i, a_i + s_i)\big)_{i=1}^I, \ \big([b_j, b_j + \sigma_j)\big)_{j=1}^J \quad \text{form a partition of } X. \tag{6}$$

Definition 3. A feasible allocation $[(c_i, a_i, s_i, Q_i)_{i=1}^I, c_L, (z_j, b_j, \sigma_j, q_j)_{j=1}^J]$ is called *Pareto optimal with J active firms* if all $z_j > 0$ and there is no other feasible allocation $[(c_i', s_i', a_i', Q_i')_{i=1}^I, c_L', (z_j', b_j', \sigma_j', q_j')_{j=1}^J]$ with all $z_j' > 0$ such that $c_L' \geqslant c_L$ and for each $i = 1, \dots, I$, $u(c_i', s_i') \geqslant u(c_i, s_i)$, with a strict inequality holding for at least one of these relations.

It is important to note that this concept of efficiency does not allow entry or exit of firms.

Definition 4. A *competitive equilibrium* consists of a feasible allocation $[(c_i, a_i, s_i, Q_i)_{i=1}^I$, $c_L, (z_j, b_j, \sigma_j, q_j)_{j=1}^J]$, an integrable land price function $p : X \to \mathbf{R}$, a vector of profits $\pi \in \mathbf{R}^J$ and a wage $w \in \mathbf{R}$ (the freely mobile composite consumption commodity is taken to be numeraire), such that

$$c_L = \int_0^l p(x)\, dm(x) + \sum_{j=1}^J \pi_j, \tag{7}$$

$$(c_i, a_i, s_i, Q_i) \quad \text{solves (1)} \quad \text{for } i = 1, \dots, I, \tag{8}$$

$$(\pi_j, z_j, b_j, \sigma_j, q_j) \quad \text{solves (2)} \quad \text{for } j = 1, \dots, J. \tag{9}$$

The allocation component of a competitive equilibrium is called an *equilibrium allocation*.

This equilibrium concept does allow firms to shut down, but does not allow entry beyond J firms.[9] In equilibrium, firm profits are non-negative (and possibly positive).

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[8] Condition (5) requires that all people work. Strictly speaking, this is not necessary. However, since we will assume that there is no disutility of work and utility is increasing in consumption, (5) will hold in equilibrium. Also, condition (6) requires that all land is used. This will hold in equilibrium since we will assume that utility is increasing in land consumption.

[9] Debreu [16] has a similar feature, but there it is less innocent, for he assumes non-increasing returns to scale, which favors small-scale production and unlimited entry. Our inclusion of a fixed cost puts a bound on the number of firms.

3. Existence of equilibrium with one producer and one consumer

Due to the discreteness and nonconvexities inherent in the model,[10] we prove that an equilibrium exists by actually finding some.

In this section we examine the following set of examples. Let $I = 1$ and $J = 1$, and for notational simplicity, drop the subscripts referring to agents. We will find particular equilibria (others exist as well) with two types of rent densities: continuous and discontinuous.

Definition 5. We say that the *functional form restriction* holds when utility satisfies the following condition: $u(c, s) = c + \alpha \cdot \ln(s)$, $\alpha > 0$.

Next, let us give bounds on exogenous parameters for continuous equilibrium rent densities.

Definition 6. We say that the *parameter restrictions for continuous equilibrium rent densities* hold when the following conditions are met: $l \geqslant 2.87$, $0 < f < \phi_c(\alpha, l)$, $\beta \geqslant B_c(\alpha, l)$, $t \geqslant \tau_c(\alpha, l)$, where the functions ϕ_c, B_c, and τ_c, all mapping $\mathbf{R}^2$ into $\mathbf{R}$, are defined in Appendix A.

In essence, what is needed is that total land $l \geqslant 2.87$, fixed cost f is small relative to the marginal utility of land (α), the marginal product (β) is large relative to α, and commuting cost (t) is large relative to α. Clearly, these restrictions represent a set of parameters with nonempty interior.

The fixed cost must be small here to guarantee that the producer can be subsidized on its parcel so that the fixed cost is covered but the consumer will not encroach. If the fixed cost is high, then a low price of land on the producer parcel covering the fixed cost will induce the consumer to encroach.

Theorem 1. *Under the functional form restriction and the parameter restrictions for continuous equilibrium rent densities, there exists an equilibrium.*

For proof, see Appendix A.

Figure 1 provides a picture of the equilibrium. The horizontal axis represents the location space, while the vertical axis is used for the land price density (in dollars per foot or inch). The horizontal axis is located not at height zero, but at height $\alpha/(l-1)$, the equilibrium marginal utility of land for the consumer. The firm is located on the parcel $[0,1)$ while the consumer buys the remainder of the land. The shaded area is the implicit subsidy from the landlord to the producer, in dollars. The price density is in fact the minimum of two curves representing marginal willingness to pay for land of the consumer over $(0, l-1)$ and $(1, l)$ (starting from the consumer's right and left endpoints, respectively).

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[10] As described in Berliant and Fujita [9], demand (and in the present model, supply) correspondences are not convex-valued. In fact, the contract curve in the pure exchange model is disconnected; see Fig. 2 of that paper.
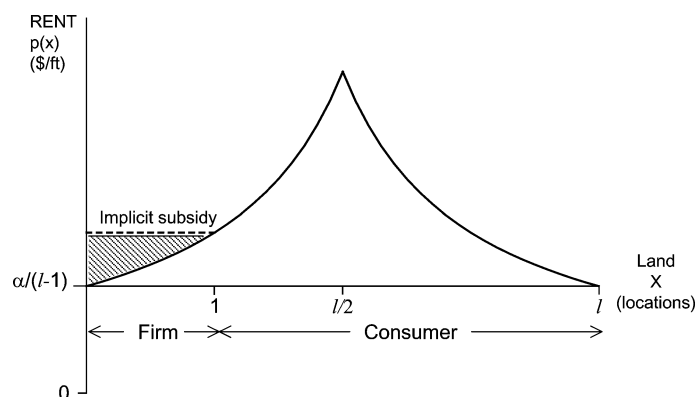
Fig. 1. Continuous equilibrium rent density.

Heuristically, this is an equilibrium for the following reasons. Regarding the consumer, the first order conditions for problem (1) tell us that the price of the marginal piece of land purchased on the end farthest from the firm must be equal to the marginal willingness to pay for land, or $p(a + s) = \alpha/s$, and that the price of marginal piece of land purchased closest to the firm, $p(a)$, must be between the marginal willingness to pay for land generally, α/s, and the marginal willingness to pay for additional land plus the associated reduction in commuting cost, $\alpha/s + t$, from having the front of the parcel closer to the firm. The latter condition arises because marginal commuting cost drops discontinuously from t to 0 as the consumer becomes adjacent to the firm. With our quasi-linear utility function, these first-order conditions are satisfied by the parcel $[1, l)$. Notice that if t is not large enough, then this last condition might not hold; that is why there is a parameter restriction on t. Regarding the firm, profits are location independent, so the firm simply wants to buy a parcel that is cheapest per unit of land purchased. Given the price density, either the left endpoint is at 0 or the right endpoint is at 1. Optimization over the amount of land used by the firm yields a price equals marginal revenue product condition. Given an equilibrated wage, this will occur when the firm uses either $[0, 1)$ or $[l - 1, l)$. Symmetry of the price density around $l/2$ is important for showing that the consumer and firm would not want to inhabit the same parcel.

Land payments follow the p contour, but land use by agents is adjusted in response to the marginal price paid for an extra unit of land. While the firm would incur a loss if it had to pay this marginal price for each unit of land it uses, lower inframarginal prices in $[0, 1)$ can generate zero profit.[11] Notice that if fixed cost f is too large, the implicit subsidy cannot cover it. That is why there is a parameter restriction on f.

Next we shall study another class of equilibria for this same model, one that is motivated by the observation that marginal commuting cost is discontinuous when the consumer and producer are adjacent. Marginal commuting cost drops from t to zero when the consumer and producer touch, thus allowing a discontinuity in land rent at the boundary.

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[11] The same kind of subsidy could apply to consumers, but it is not relevant for them. There is no analog of the non-negative profit condition for consumers, whereas this is a participation constraint for producers in our model.

Definition 7. We say that the *parameter restrictions for discontinuous equilibrium rent densities* hold when the following conditions are met: $l \geqslant 3.19$, $0 < f \leqslant \phi_d(\alpha, l)$, $\beta \geqslant B_d(\alpha, l)$, $t \geqslant \tau_d(\alpha, l)$, where the functions ϕ_d, B_d, and τ_d, all mapping $\mathbf{R}^2$ into $\mathbf{R}$, are defined in Appendix A.

Once again, total land (l) needs to be large enough, while fixed cost (f) must be small relative to the marginal utility of land (α), the marginal product (β) must be large relative to α, and commuting cost (t) must also be large relative to α. Again, these restrictions represent a set of parameters with nonempty interior.

Theorem 2. *Under the functional form restrictions and the parameter restrictions for discontinuous equilibrium rent densities, there exists an equilibrium.*

For proof, see Appendix A.

Figure 2 provides a picture of the equilibrium. The horizontal axis represents the location space, while the vertical axis is used for the land price density (in dollars per foot). The horizontal axis is located not at height zero, but at height $\alpha/(l-1)$, the equilibrium marginal utility of land for the consumer. The firm is located on the parcel $[0, 1)$ while the consumer buys the remainder of the land. The shaded area is the implicit subsidy from the landlord to the producer, in dollars.

The intuition for why Fig. 2 represents an equilibrium is very much the same as the intuition for why Fig. 1 represents an equilibrium. The discontinuity in rent is admissible for the following reasons. From the viewpoint of the consumer, it does not induce further purchase of land, since at 1 (and to the left of 1), price is just equal to marginal willingness to pay, $\alpha/(l-1)$, and the marginal reduction in commuting cost from moving left of 1 is nil. From the viewpoint of the firm, expansion of its parcel to the right of 1 means less profit, since the marginal revenue product of land is equal to its price at 1.
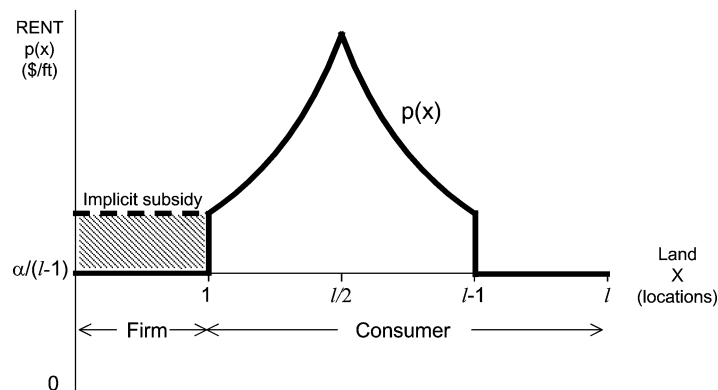

Fig. 2. Discontinuous equilibrium rent density.

4. Existence of equilibrium with two producers and many consumers

This generalization of the model is not as easy as it may appear. In this section, first we will examine the natural extension of the model to multiple producers and explain what goes wrong with existence of equilibrium. Then we will make a modification so as to obtain existence of equilibrium.

Consider a model with one producer and an even number, say $2I$, of consumers. Let us examine a continuous rent density equilibrium. To keep the model as close as possible to the one in the last section, let us change the technology to $g(\sigma, q) = \beta \cdot \min(\sigma, q/I) - f$, and let $X = [-l + 1, l]$. One way to construct a continuous rent density is illustrated in Fig. 3. In the end, this figure will *not* represent an equilibrium. Again, the horizontal axis represents location space while the vertical axis gives the price density for land in dollars per foot. The horizontal axis is located at height $\alpha I/(l - 1)$ rather than at zero on the vertical axis. The price density is the same as in the previous section for the consumer to the right of the firm. We replicate the same density for the consumer to the left of the firm. This necessitates an alteration of the density on the firm's parcel, due to the presence of land to the left of the firm that it would want to buy unless the price were raised (this is justified by the first-order condition for firm optimization with respect to b). Thus, we take the maximum of these two price densities. However, land at the extreme left and extreme right in X is cheapest under this new density, so the firm would move out to an extreme. To prevent this, we must raise the price of land in the extremes by replicating a shifted price density once again, and taking the maximum of all price densities. This will violate the first order conditions for the consumers, which state that the price of the edge of a parcel closer to the firm must be t higher than the edge further away from the firm (as in Berliant and Fujita [9]). This statement does not apply to the innermost two consumers, since there is a discontinuity in their marginal commuting cost at zero distance; there is no such discontinuity for consumers not adjacent to the firm, so this statement must apply
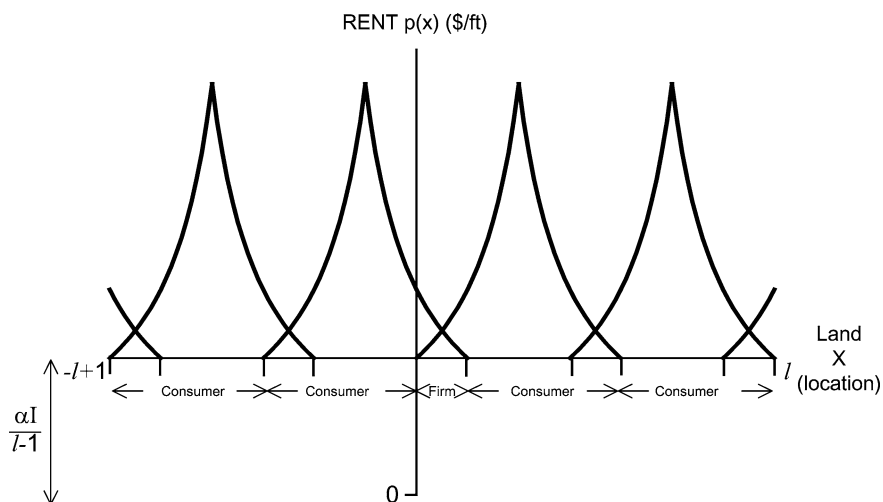

Fig. 3. Multiple consumers—continuous rent density.

to them. Moreover, given that the price density on each consumer parcel is the same, the total cost of each consumer parcel is the same, so why would any consumer choose to live on a parcel not adjacent to the firm? They would pay the same total land rent, but incur a higher commuting cost further out, thus attaining a lower level of utility. Figure 3 does not represent an equilibrium.

So how can we solve this problem and obtain an equilibrium? The answer to this question lies in noticing that the problem we have is overconstrained. We are asking too much of the rent density, in that it reflects differences in commuting cost among parcels as stated above (essentially the Mills [30]–Muth [31] condition for our model),[12] but at the same time, reflects the fact that the profit function only accounts for the cost and not the location of the parcel, so the producer will always choose the cost minimizing one. In other words, consumer optimization requires that rent decreases as distance from a producer increases, to compensate for commuting costs, while the producer will always find the lowest cost parcel, located as far as possible from its current spot.

If prices are low on the producer parcel, then consumers will move there to reduce commuting cost. If prices are low on consumer parcels distant from the producers to compensate for commuting cost, then producers will move there to reduce land cost. Equilibrium is not likely to exist. This is in essence the problem discovered by Koopmans and Beckmann [25] in their investigation of the quadratic assignment problem.[13] Although their model is different from ours, this kind of problem pertaining to existence of equilibrium arises in most location models where all agents and resources are mobile.

We must specify out-of-equilibrium commuting costs properly. In the pure exchange version of the Alonso model, the location to which consumers commute, the central business district or CBD, is given and occupies no land. Commuting cost is given by the "front location" or "front door" (closest point) distance from the consumer's parcel to the CBD. See Asami et al. [6] for elaboration. However, if a producer (or the CBD) occupies space, it is unclear, especially out of equilibrium, where the consumer must commute to. For instance, if the consumer decides to buy a subset of the parcel used by a producer, clearly a disequilibrium situation, what is its commuting distance and cost? This must be specified, even out of equilibrium, in order to verify whether a particular situation represents an equilibrium or not.

We assume that if a consumer outbids a producer, he or she can no longer work at that location, since the producer will no longer be there. Consumers and producers remain price takers; this is simply a specification of disequilibrium commuting costs. Formally, it amounts to defining commuting distance for consumer i to firm j as

$$T_i^j(a_i, s_i, b_j, \sigma_j) = \begin{cases} \inf_{x \in (a_i, a_i+s_i),\ y \in (b_j, b_j+\sigma_j)} t\|x - y\| \\ \qquad \text{if } (a_i, a_i + s_i) \cap (b_j, b_j + \sigma_j) = \emptyset, \\ \infty \quad \text{if } (a_i, a_i + s_i) \cap (b_j, b_j + \sigma_j) \neq \emptyset. \end{cases}$$

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[12] See, for instance, Fujita [20, p. 25, Eq. (2.37)] for a nice statement and explanation.

[13] The quadratic assignment problem is distinct from, but related to, the linear assignment problem (or one-sided matching problem) that is generally more familiar to economists. The quadratic assignment model allows flows of (intermediate) goods between agents, at some cost.
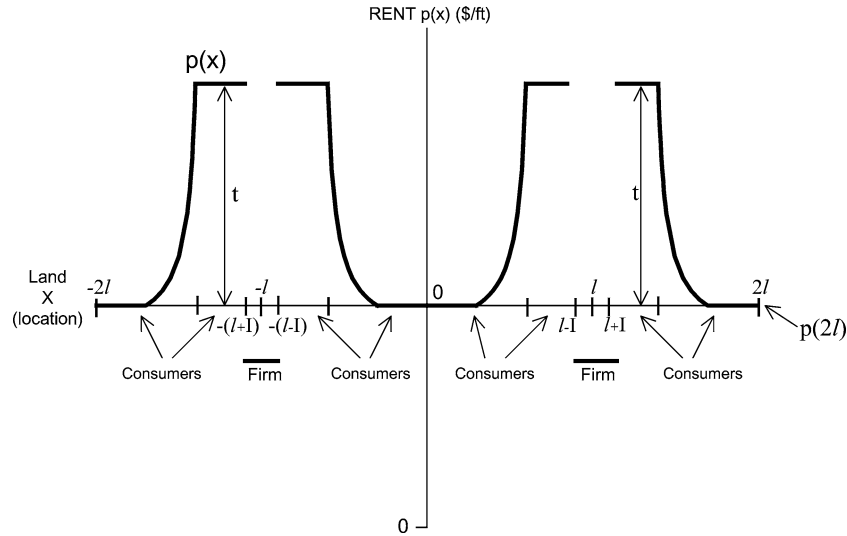
Fig. 4. Equilibrium with multiple consumers and two firms.

Commuting cost is defined to be $\min_j T_i^j(a_i, s_i, B)$, analogous to the Alonso model. We say that *commuting cost satisfies the functional form restriction* when this commuting cost function is used.[14] Notice that this commuting cost function is not upper semicontinuous in consumer location; it can drop discontinuously as the intersection of consumer and producer parcels tends to the empty set.

Figure 4 illustrates what an equilibrium will look like. The horizontal axis represents the location space $X = [-2l, 2l]$, while the vertical axis is used for the land price density (in dollars per foot). The horizontal axis is located not at height zero, but at height $p(2l)$, the equilibrium marginal utility of land for the consumers located farthest from a firm. Equilibrium configurations consist of individual producers surrounded by commuting consumers. This configuration involves agglomeration around a producer, essentially a company town. Notice that parcels get cheaper as we move out away from a firm. This is necessary in equilibrium in order to compensate for the increased cost of commuting as distance from the firm increases, for otherwise nobody would live in the hinterlands. Notice also that we can do this while still making the firm's parcel the cheapest per unit cost of land, so the firm has no incentive to move. The modification of the commuting cost function implies that no consumer will encroach on a producer's parcel, since encroachment means that the consumer must commute to the next closest producer, requiring a large jump in expenditure on commuting. Thus, the commuting cost deters consumer encroachment into a firm's parcel, and the low price of land on a firm's parcel keeps the firm there.

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[14] We intend to attack the Koopmans–Beckmann quadratic assignment problem head on, using the same modification of out-of-equilibrium transport costs that we have used here for commuting costs. If an agent wants to cohabit a parcel with another, then it must go elsewhere for supplies (or more generally, transactions). In closing, we note that the quadratic programming disease is present in many location models.

There will be some restrictions on the parameters. The equilibrium will have the same pattern as equilibrium in the Alonso model, that consumers with higher wages live further from the firm and buy more land. As in Berliant and Fujita [9], we try to find equilibrium allocations that are Pareto optimal and use the property that richer consumers purchase more land and are located farther from the producer (otherwise we can switch positions of the consumers, save on commuting costs, and create a Pareto improvement).[15] For simplicity, we shall only examine the case when all consumers are identical.

To make notation simpler, let $X = [-2l, 2l]$. We focus on the part of the economy to the right of 0 in X; the part to the left will be symmetric. We return to using the production function $g(\sigma, q) = \beta \cdot \min(\sigma, q) - f$. There are $4I$ consumers. In contrast with the assumptions of the preceding section, we allow a general utility function. The utility function of every consumer is $u(c, s)$, where $u : \mathbf{R}_+^2 \to \mathbf{R}$ satisfies the following conditions, the first three of which are adapted from Berliant and Fujita [9, Assumption 1]. Let $c = C(s, u)$ define the indifference curve at utility level u and denote a partial derivative by a subscript. As is standard, the implicit function theorem gives us that $-C_s(s, u) = (u_s/u_c)(c, s)$. This is the marginal rate of substitution of composite good for land, or the marginal willingness to pay for land.

Definition 8. A utility function u is said to be *well-behaved* if it satisfies the following:

(i) On $\mathbf{R}_{++}^2$, u is twice continuously differentiable, strictly quasi-concave, $u_c > 0$, and $u_s > 0$.
(ii) No indifference curve intersecting $\mathbf{R}_{++}^2$ cuts an axis, and every indifference curve intersecting $\mathbf{R}_{++}^2$ has the c-axis as an asymptote.
(iii) Lot (or land) size s is a normal good on $\mathbf{R}_{++}^2$.
(iv) The composite consumption commodity is a normal good on $\mathbf{R}_{++}^2$.
(v) For each fixed u, $-C_s(s, u)$ is a convex function of s.
(vi) For each fixed $s > 0$, $C_{ss}(s, u)$ is a nondecreasing function of u.

Cobb–Douglas utilities are an example.

Definition 9. The *parameter restrictions for two producers* are said to be satisfied if the following hold: $I \geqslant 2$, $l \geqslant 2I^2 + I$, $0 < f/\beta \leqslant (16/17)I$, $t/\beta \geqslant 9/17$. Finally, the marginal willingness to pay for land satisfies the following inequality at a particular (given) allocation $(\bar{c}, \bar{s}) > (0, 0)$ (specified in Appendix A): $(u_s/u_c)(\bar{c}, \bar{s}) > \theta(I, l, \beta, f, t)$, where the function $\theta : \mathbf{R}^5 \to \mathbf{R}$ is given in Appendix A.

For example, a CES utility function will satisfy the last inequality if parameters are chosen appropriately.

These parameter restrictions imply that the total land available (l) is large relative to the number of consumers and that marginal product (β) is large relative to fixed costs (or that

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[15] If land is a normal good, consumers with higher wages and thus more income will purchase more land. Although land is not strictly normal in the example we considered in Section 3, it is weakly normal in the sense that the income derivative of demand for land is zero, so the argument applies.

the number of consumers is large relative to fixed costs) but small relative to commuting costs. The condition on marginal willingness to pay for land at a particular bundle implies that one consumer's land consumption cannot become too small relative to another's.

Theorem 3. *If the utility function is well-behaved, commuting cost satisfies the functional form restriction, and the parameter restrictions for two producers hold, then there exists an equilibrium.*

For proof, see Appendix A. Figure 4 provides a picture of the equilibrium, and was explained earlier in this section.

The strategy of the proof is as follows. Guess that the firms' parcels are $[-(l + I), -(l - I)]$ and $[l - I, l + I]$. Then we fix a wage rate, and solve the consumer equilibrium problem on the parcels not occupied by firms, exploiting the results of Berliant and Fujita [9] to construct an equilibrium. We set the firm land price lower than the lowest consumer price, the difference depending only on fixed costs, total land available, and the number of consumers. Then we set up the zero profit condition of the firm in equilibrium, and find a wage rate that solves it. This wage rate, the implied rent density, the allocation of land, and the allocation of consumption good form an equilibrium. The hard part of the proof is to show that no consumer would intrude on a firm's parcel, and vice versa.

The details of the proof can be found in Appendix A.

5. The First Welfare Theorem

In this section we show that an equilibrium allocation *can* be first best, though it is not necessarily first best. There are two reasons an equilibrium allocation might not be first best in this model. First, the *entry or exit* of a firm causes an externality in that the firm does not account for the changes in commuting cost to consumers as a consequence of its decision. Second, the *location* decision of a firm causes an externality in that the firm does not account for the changes in commuting costs of consumers as a consequence of its decision. We can characterize equilibrium allocations that are optimal in the second sense, namely with a fixed number of firms.

For notational convenience, in this section we use $X = [-2l, 2l]$ as the totality of land available. The production function remains $g(\sigma, q) = \beta \cdot \min(\sigma, q) - f$ and the number of consumers remains I.

Definition 10. An allocation $[(c_i, a_i, s_i, Q_i)_{i=1}^I, c_L, (z_j, b_j, \sigma_j, q_j)_{j=1}^J]$ is called *symmetric in production* if

(i) the number of consumers commuting to a firm from the left and right are equal and the same for all firms; that is, for all j, the cardinality of the sets $\{i \mid 1 \leqslant i \leqslant I, Q_i^j = 1, a_i \leqslant b_j\}$ and $\{i \mid 1 \leqslant i \leqslant I, Q_i^j = 1, a_i \geqslant b_j\}$ is the same and independent of j, and

(ii) the midpoints of the firm land parcels are evenly dispersed; that is, if the numbering of firms is such that the midpoints of their parcels are ordered from left to right, then $b_j + \sigma_j/2 = -2l + 2l/J + 4(j - 1)l/J$.

Notice that by the first requirement, $I/(2J)$ must be integer.

Due to the form of the production function, for all producers j, land usage is $\sigma_j = I/J$ at any equilibrium allocation that is symmetric in production. If we wish to examine the efficiency properties of an equilibrium allocation in which a firm is shut down, then we can simply reduce J.

Theorem 4. *Suppose that the utility function u is well-behaved. Fix any equilibrium that is symmetric in production, and set J to be the number of firms j with $z_j > 0$ (eliminating the firms that are shut down). Then the equilibrium allocation is Pareto optimal with J active firms.*

For proof, see Appendix A.

The purpose of this result is to cover the situation studied in Section 4. The result can easily be extended to the situations discussed in Section 3, where $X = [0, l)$, $I = 1$, and $J = 1$, or more generally to cases where $I/(2J)$ is not integer. However, the benefit of additional generality from such results is exceeded by the cost of additional complexity that is introduced.

Notice that no agent has as their objective the minimization over J of the fixed cost of J firms plus total commuting cost, $fJ + \sum_{i=1}^{I} \min_j T_i^j(a_i^J, s_i^J, B^J)$, where $(a_1^J, s_1^J, \ldots, a_I^J, s_I^J, B^J)$ is an equilibrium parcel configuration with J active firms. The landlord comes closest to having this as an objective (through maximization of land rent); an equilibrium concept in which the landlord implicitly chooses the number of active firms by choosing the rent density could be formulated, but the objective is still not quite the same as minimization of fixed costs plus aggregate commuting cost. Since J is not chosen by an agent who accounts for the externality, one cannot in general expect equilibrium allocations to result in an optimal number of active firms. This explains the notion of efficiency that is used here, which is conditional on J active firms. If J happens to minimize fixed cost plus aggregate commuting cost, then Theorem 4 implies that an equilibrium allocation that is symmetric in production is first best.

6. Conclusions and extensions

Using some classes of examples, we have examined how land can reconcile increasing returns and perfect competition in the following sense. In a model without location, production of a commodity using a technology requiring a fixed cost followed by constant returns to scale will imply that only one firm producing this good will operate in an efficient allocation. However, in a spatial model with commuting cost, such as the one examined here, there is a trade-off between returns to scale and the cost of accessing a firm, thus limiting the extent of the market served by any single firm, and therefore allowing multiple active firms in an efficient allocation. A perfectly competitive equilibrium can result in a land price scheme that limits firm size optimally and provides a subsidy to active firms consistent with efficiency.

The numbers of firms and consumers can be made large by replicating the example of Section 4.

The questions we have studied seem important not only in the theory of industrial organization, in that government intervention in markets for goods produced under an increasing returns to scale technology may not be justified, but also in the theory of spatial economics. For example, we can separate results due to imperfect competition from those due to the presence of location in models. These questions are of central interest to urban economics and location theory as well. The Spatial Impossibility Theorem of Starrett [44], as interpreted by Fujita [18], tells us that some assumption of neoclassical economics must not hold if we are to generate equilibrium models of agglomeration. Here we have used increasing returns and perfect competition, but we are able to generate agglomeration and factory towns in equilibrium without imperfect competition. Unlike much of the other work on agglomeration, our equilibrium configurations can be first best.[16]

Here we have assumed perfect competition, but have not justified this assumption formally. The latter should be the subject of future work; the tests of Gretsky et al. [24] for perfect competition should be useful.

One testable implication derived from the model is that the unit land price of a firm's parcel should be low relative to the unit price of residential land surrounding the producer. Of course, the hazards involved in testing this hypothesis include the difficulty in separating the value of land from structures as well as zoning laws.

Another issue of interest is the conjecture that, in both this model and the simpler Alonso exchange model, even though equilibria exist and equilibrium allocations are Pareto optimal (see Berliant and Fujita [9] for the exchange case), the core can be empty. Thus far, we have a quasi-linear example (see Section 3) where the emptiness or non-emptiness of the core depends on endowments. We intend to look at this more generally, and examine the implications for core convergence.

Acknowledgments

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[16] A referee has suggested that, as a further extension, the assumption that consumers have no intrinsic preference for location be relaxed as follows. Consumers have single peaked preferences over location with bliss points uniformly distributed over front locations in X. In general, heterogeneity in consumer utilities or endowments such as locational preference allows *more* room for existence of equilibrium, since equal utility conditions no longer need to hold in equilibrium. However, heterogeneity also makes the calculations in the analysis much messier. Probably extensions such as this one will have to wait for different techniques of proof.

Appendix A

A.1. Parameter restrictions for continuous equilibrium price densities

$$\phi_c(\alpha, l) = \alpha/[l-2] - \alpha \cdot \ln[(l-1)/(l-2)],$$
$$B_c(\alpha, l) = \alpha/(l-2) + 2\alpha \cdot \ln[(l-1)/(l/2-1)] - \alpha \cdot \ln[(l-1)/(l-2)],$$
$$\tau_c(\alpha, l) = 2\alpha[1/(l-2) - 1/l].$$

It is easy to see that the functions ϕ_c and B_c are positive.

Proof of Theorem 1. Let $p(x) = \alpha/(l-x-1)$ for $x \leqslant l/2$, $p(x) = \alpha/(x-1)$ for $x \geqslant l/2$, $b = 0$, $\sigma = 1$, $q = 1$, $z = \beta - f$, $a = 1$, $s = l-1$, $Q = [1]$, $w = \beta - \alpha/(l-2)$,

$$\pi = \beta - f - w - \alpha \cdot \ln[(l-1)/(l-2)],$$
$$c = w - \{2\alpha \cdot \ln[(l-1)/(l/2-1)] - \alpha \cdot \ln[(l-1)/(l-2)]\}$$

(which is non-negative by the assumption on β), and $c_L = 2\alpha \cdot \ln((l-1)/(l/2-1)) + \pi$. We claim that this is an equilibrium. Figure 1 provides a sketch of the price density.

First, we verify that this is indeed a feasible allocation. To verify (3), note that commuting cost is zero in this allocation, and calculate:

$$\begin{aligned} c + c_L &= w - \left(2\alpha \cdot \ln\frac{l-1}{l/2-1} - \alpha \cdot \ln\frac{l-1}{l-2}\right) + 2\alpha \cdot \ln\frac{l-1}{l/2-1} + \beta - f \\ &\quad - w - \alpha \cdot \ln\frac{l-1}{l-2} \\ &= \beta - f = z. \end{aligned}$$

Conditions (4) and (5) are obvious. Finally, note that $[0, 1)$, $[1, l)$ is indeed a partition of X, so (6) holds.

Regarding the equilibrium conditions (7)–(9), (7) can be verified simply by calculating the total area under the price density, $2\alpha \cdot \ln[(l-1)/((l/2)-1)]$, and adding to it profits π.

Problem (1) can be written as the following unconstrained optimization problem by substituting the budget constraint for c:

$$\max_{a,s} \alpha \cdot \ln(s) + w - \int_a^{a+s} p(x)\, dm(x) - t \cdot \max(0, a-1).$$

The first-order condition with respect to s is $p(a+s) = \alpha/s$; this is verified for our price density at $a = 1$ and $s = l-1$. The first-order condition with respect to a is $p(a) - p(a+s) = t$ if $a > 1$, $p(a) - p(a+s) \in [0, t]$ if $a = 1$, $p(a) - p(a+s) = 0$ if $a < 1$. This is an interesting and important fact. Notice first that if $a = 1$, the parameter restriction on t implies $p(a) - p(a+s) = \alpha/(l-2) - \alpha/(l-1) < 2\alpha[1/(l-2) - 1/l] \leqslant t$, so our equilibrium satisfies the first-order condition. Second, this first-order condition is a result of the assumption that closest point distance is all that matters when computing commuting cost, so discontinuous *marginal* commuting cost is the consequence. Total commuting cost is continuous.

Regarding second-order conditions for the consumer, it is rather evident that the consumer cannot do better by decreasing its parcel size to the right of $l/2$, since the rent curve is equal to the marginal willingness to pay for land of the consumer with left endpoint at 1; if the left endpoint is greater than 1, then marginal willingness to pay exceeds price. For points $x \in (1, l/2]$, we must prove that marginal utility of land exceeds price less the reduction in commuting cost from purchasing additional land closer to the producer. Marginal utility is $\alpha/(l - x)$, while price is $\alpha/(l - 1 - x)$ and commuting cost is t. Thus, for $x \in (1, l/2]$, we must show that $\alpha/(l - x) \geqslant \alpha/(l - 1 - x) - t$. The parameter restriction on t is $t \geqslant 2 \cdot \alpha(1/(l - 2) - 1/l)$, so $t \geqslant \alpha[1/(l/2 - 1) - 1/(l/2)]$ and $\alpha/(l - x) \geqslant \alpha/(l - 1 - x) - t$ at $x = l/2$. Since

$$\frac{\partial}{\partial x}[\alpha/(l - x) - \alpha/(l - 1 - x) + t] = \alpha\left[1/(l - x)^2 - 1/(l - 1 - x)^2\right] < 0,$$

$\alpha/(l - x) \geqslant \alpha/(l - 1 - x) - t$ for all $x \in (1, l/2]$. The consumer cannot do better by increasing its parcel size (starting from $[1, l)$) since for larger parcels, the rent curve $\alpha/(l - x - 1)$ is greater than the marginal willingness to pay for land $\alpha/(l - x)$. Due to the symmetry of the rent curve, the consumer cannot do better by owning a parcel containing $\{0\}$ rather than $\{l\}$. Thus, the equilibrium allocation solves (1) for the consumer.

With regard to the firm, notice that optimization will imply that $q = \sigma$ and optimization problem (2) reduces to

$$\max_{b,\sigma} \beta \cdot \sigma - f - \int_{b}^{b+\sigma} p(x) \, dm(x) - w \cdot \sigma.$$

The first-order condition with respect to σ is $\beta - p(b + \sigma) - w = 0$, and w was chosen to satisfy this equality for $b = 0$ and $\sigma = 1$. The first-order condition with respect to b is $p(b) = p(b + \sigma)$,[17] which can either be ignored since the producer hits the land boundary at zero, or we can set $p(0) = \alpha/(l - 2)$, altering p on a set of measure zero.

Turning next to second-order conditions for the firm, notice first that if the firm uses a parcel of any size, it is indifferent about its location, so it will choose one of the cheapest parcels, and $[0, \sigma)$ is among these. The first-order condition with respect to σ will imply that it will choose $\sigma = 1$. Beyond this, up to $\sigma = l/2$, the marginal cost of land exceeds the marginal benefit net of labor cost. If the firm can make higher profits from expanding the scale of its operations beyond 1, then given the production function and the price density, it will make higher profits when $b = 0$ and $\sigma = l$. Profits from such a production plan are given by

$$\beta \cdot l - f - w \cdot l - 2 \cdot \alpha \cdot \int_{l/2}^{l} 1/(x - 1) \, dm(x). \tag{10}$$

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[17] This reflects the location independence of the production function.

Profits from the equilibrium production plan are given by

$$\beta - f - w - \alpha \cdot \int_{l-1}^{l} 1/(x-1)\, dm(x). \tag{11}$$

Following some calculations, it can be shown that (11) always exceeds (10) if $[(l-1)/(l-2)] \leqslant 2 \cdot \ln(2) + \ln[(l-1)/(l-2)]$ or, as assumed above, $l \geqslant 2.87$.

Finally, it is necessary to show that (11) is non-negative, in order to ensure that the producer will not exit. Again, following some calculations, the assumption that $f \leqslant \alpha/[l-2] - \alpha \cdot \ln[(l-1)/(l-2)]$ implies that (11) is always non-negative. $\quad\square$

A.2. Parameter restrictions for discontinuous equilibrium price densities

$$\phi_d(\alpha, l) = \alpha \cdot [1/(l-2) - 1/(l-1)],$$

$$B_d(\alpha, l) = \alpha/(l-2) + \alpha/(l-1) + 2\alpha \cdot \ln[(l-2)/(l/2-1)],$$

$$\tau_d(\alpha, l) = 2 \cdot \alpha[1/(l-2) - 1/l].$$

Proof of Theorem 2. Let $p(x) = \alpha/(l - x - 1)$ for $1 \leqslant x \leqslant l/2$, $p(x) = \alpha/(x-1)$ for $l - 1 \geqslant x \geqslant l/2$, $p(x) = \alpha/(l-1)$ for $0 \leqslant x < 1$, $p(x) = \alpha/(l-1)$ for $l - 1 < x \leqslant l$, $b = 0$, $\sigma = 1$, $q = 1$, $z = \beta - f$, $a = 1$, $s = l - 1$, $Q = [1]$, $w = \beta - \alpha/(l-2)$, $\pi = \beta - f - w - \alpha/(l-1)$, $c = w - \{\alpha/(l-1) + 2\alpha \cdot \ln[(l-2)/(l/2-1)]\}$ (which is non-negative by the assumption on β), and $c_L = 2\alpha/(l-1) + 2\alpha \cdot \ln[(l-2)/(l/2-1)] + \pi$. We claim that this is an equilibrium. Figure 2 provides a sketch of the price density.

First, we verify that this is indeed a feasible allocation. To verify (3), note that commuting cost is zero in this allocation and calculate:

$$\begin{aligned}
c + c_L &= w - \left(\frac{\alpha}{l-1} + 2\alpha \cdot \ln \frac{l-2}{l/2-1}\right) + \frac{2\alpha}{l-1} + 2\alpha \cdot \ln \frac{l-2}{l/2-1} + \beta - f \\
&\quad - w - \frac{\alpha}{l-1} \\
&= \beta - f = z.
\end{aligned}$$

Verification of Eqs. (4) and (5) is obvious. Finally, note that $[0, 1), [1, l)$ is indeed a partition of X, so (6) holds.

Regarding the equilibrium conditions (7)–(9), (7) can be verified simply by calculating the total area under the price density, $2\alpha/(l-1) + 2\alpha \cdot \ln[(l-2)/(l/2-1)]$, and adding to it profits π.

As the reader might suspect, the remainder of the proof that the specified discontinuous rent density and allocation is in fact an equilibrium is quite analogous to the proof for continuous equilibrium rent densities, so we shall not bother to repeat it here. The proof that equilibrium profits are larger than profits using all land involves solving a quadratic equation, the largest root of which is approximately 3.19. $\quad\square$

A.3. Parameter restrictions for two producers

Let (c', s') solve $\max_{c,s} u(c, s)$ subject to $c + p's \leqslant w'$ for $w' = (\beta + f/(4(l - I)))/(1 + I/(l - I))$ and $p' = \beta + f/(4(l - I)) + (I - 1)t$. Let $u^* = u(\beta + f/(4(l - I)), (l - I)/I)$. Then specify

$$\bar{c} = \min\left\{\beta - \frac{f}{2I} - \frac{f(l/(2I) - 1/4)}{I} - \left(1 - \frac{1}{I^2}\right)(l - I)t,\right.$$
$$\left.\frac{f[l/(2I) - 1/4](l - I - I\bar{s})}{(l - I)(I - 1)}\right\}$$

and

$$\bar{s} = \max\left\{\frac{f(2l/I - 1)}{(l - I)(I - 1)t}, \frac{l - I}{I} - \frac{I - 1}{2}\frac{t}{C_{ss}(s', u^*)}\right\}.$$

$\theta(I, l, \beta, f, t) = \beta + f/[4(l - I)] + (I - 1)t$. The expressions are positive. $\bar{s} > 0$ due to the assumption on l. $\theta > 0$ by the assumption on f/β. $\bar{c} > 0$ because $\bar{s} < (l - I)/I$. To see this, consider the first expression in the definition of $\bar{s}$. It is less than $(l - I)/I$ due to the assumptions on I, l and $f/t \leqslant (16/9)I$. The second expression is obviously less than $(l - I)/I$.

Proof of Theorem 3. We begin by fixing w, the wage rate, in $[0, \beta + f/[4(l - I)]]$. Apply Proposition 4 of Berliant and Fujita [9] to the exchange economy where consumers $i = 1, \ldots, I$ have an endowment of consumption good w and land is limited to the interval $(l + I, 2l]$, to obtain an equilibrium price density $p_w(x)$, where $p_w(2l)$ is uniquely determined (and is the same for all equilibria). Using the assumption that land is a normal good, $p_w(2l)$ is increasing in w. Using upper hemi-continuity of the equilibrium correspondence of the exchange economy in w, $p_w(2l)$ is continuous in w. We want to solve

$$\beta - w - \frac{f}{2I} - p_w(2l) + f \cdot \frac{l/(2I) - 1/4}{l - I} = 0 \tag{12}$$

on $0 \leqslant w \leqslant \beta + f/[4(l - I)]$. This will be the zero-profit condition for the firms (with $p_w(2l) - f[l/(2I) - 1/4]/(l - I)$ representing rent).

As w tends to zero, $p_w(2l)$ tends to zero, so the left-hand side of (12) tends to $\beta + f/[4(l - I)]$, which is positive by assumption on l. Note that at $w = \beta + f/[4(l - I)]$, the left-hand side is $-p_{\beta+f/[4(l-I)]}(2l)$, which is non-positive. By the intermediate value theorem, there is a w^* solving the equation.

Define $p = p_{w^*}$. Mirror the allocation on the interval $(0, l - I)$. The allocations on the intervals $(-2l, -l - I)$ and $(-l + I, 0)$ are defined analogously. Let $Q_i^1 = 1$ and $Q_i^2 = 0$ if $i \leqslant 2I$. Let $Q_i^1 = 0$ and $Q_i^2 = 1$ if $i > 2I$.

For $l - I \leqslant x \leqslant l + I$, define $p(x) = p(2l) - f[l/(2I) - 1/4]/(l - I)$. The price density on the firm's parcel is less than the lowest price on any consumer's parcel.

For $0 \leqslant x \leqslant l - I$, define $p(x) = p(2l - x)$. For $-2l \leqslant x \leqslant 0$, define $p(x) = p(-x)$.

Let $b_1 = l - I$, $b_2 = -l - I$. For $j = 1, 2$ let $\sigma_j = 2I$, $q_j = 2I$, $z_j = 2I\beta - f$, $\pi_j = 0$. For consumers residing in the interval $(l + I, 2l)$,

$$c_i = w^* - \int\limits_{a_i}^{a_i + s_i} p(x)\, dm(x) - t \cdot (a_i - l - I) \geqslant 0$$

by construction of the exchange economy allocations. The consumption of other consumers is defined analogously. $c_L = \int_{-2l}^{2l} p(x)\, dm(x) \geqslant 0$.

We claim that this is an equilibrium. First we must prove that the price density on the firm's parcel is non-negative (this also ensures $c_L \geqslant 0$). This is tantamount to a lower bound on $p(2l)$, the minimal willingness to pay for land in the exchange economy equilibrium on the interval $(l + I, 2l)$. The vehicle will be the assumption on the marginal rate of substitution, but its application requires $s_1 \leqslant \bar{s}$ and $c_1 \geqslant \bar{c}$, where the parcel front locations are $a_1 \leqslant a_i \leqslant a_I$. Using the assumption that land is a normal good, $s_1 \leqslant \cdots \leqslant s_i \leqslant \cdots \leqslant s_I$, $c_1 \geqslant \cdots \geqslant c_i \geqslant \cdots \geqslant c_I$; moreover, the rent density is constant on the first parcel and decreases by t across every other parcel; see Berliant and Fujita [9]. We will also use two upper bounds. When all $s_i = (l - I)/I$, an upper bound for rent on $(l + I, 2l)$ is obtained, namely,

$$p(2l)(l - I) + \frac{(I - 1)t(l - I)}{I} + \frac{(I - 1)t(l - I)}{I} + \cdots + \frac{t(l - I)}{I}$$
$$= p(2l)(l - I) + \frac{(I - 1)(1 + I/2)t(l - I)}{I},$$

and transport cost on $(l + I, 2l)$ is maximal, namely,

$$\frac{t(l - I)}{I} + \cdots + \frac{t(I - 1)(l - I)}{I} = \frac{t(I - 1)I(l - I)}{2I}.$$

Now suppose, to the contrary, that the price density on the firm's parcel is negative, then $p(2l) < f[l/(2I) - 1/4]/(l - I)$ and by Eq. (12), $w^* > \beta - f/(2I)$. Subtracting the upper bounds for rent and transport cost, a lower bound for mean consumption is

$$\beta - \frac{f}{2I} - \frac{p(2l)(l - I)}{I} - \frac{(I - 1)(1 + I)(l - I)t}{I^2}$$
$$> \beta - \frac{f}{2I} - f \cdot \frac{l/(2I) - 1/4}{I} - \left(1 - \frac{1}{I^2}\right)(l - I)t \geqslant \bar{c},$$

by definition of $\bar{c}$. It follows that $c_1 \geqslant \bar{c}$. Next we prove $s_1 \leqslant \bar{s}$.

For this purpose we first establish a lower bound for s_1. Notice that from Eq. (12), $\beta + f/(4(l - I)) = \beta - f/(2I) + f[l/(2I) - 1/4]/(l - I) = w + p(2l)$. Also, $Iw \geqslant p(2l)(l - I)$, since all land must be purchased, so $p(2l) \leqslant Iw/(l - I)$. Substituting, $\beta + f/(4(l - I)) \leqslant w(1 + I/(l - I))$. Hence $w \geqslant w'$. Also from Eq. (12), $p(2l) \leqslant \beta + f/(4(l - I))$. Hence the price paid by consumer 1 for land is $p(2l) + (I - 1)t \leqslant \beta + f/(4(l - I)) + (I - 1)t = p'$. Since land is a normal good, $w \geqslant w'$ and $p(2l) + (I - 1)t \leqslant p'$ yield $s_1 \geqslant s'$.

Denote the equilibrium level of utility for all consumers by u. By assumption $-C_s(s, u)$ is convex, hence $C_s(s, u)$ is concave and $C_{ss}(s, u)$ is nonincreasing in s, so that the

mean value theorem implies $C_s(s_2, u) - C_s(s_1, u) \leqslant C_{ss}(s', u)(s_2 - s_1)$. But the left-hand side of this inequality is t, the drop in rent across the parcel of consumer 2. It follows that $s_2 - s_1 \geqslant t/C_{ss}(s', u) \geqslant t/C_{ss}(s', u^*)$ where $u^* = u(\beta + f/(4(l - I)), (l - I)/I) \geqslant u(c_1, s_1)$, using the assumption that C_{ss} is nondecreasing in u. In fact, this argument applies to every pair of adjacent consumers (there is nothing special about consumers 1 and 2). Thus, $s_1 \leqslant s_i - (i - 1)t/C_{ss}(s', u^*)$, so $Is_1 \leqslant l - I - I(I - 1)t/(2C_{ss}(s', u^*))$; thus $s_1 \leqslant (l - I)/I - (I - 1)t/(2C_{ss}(s', u^*))$.

Consumer 1 pays rent density $p(2l) + (I - 1)t$. This price equals the consumer's marginal willingness to pay for land which exceeds $\theta(I, l, \beta, f, t) = \beta - f/[4(l - I)] + (I - 1)t$ by assumption on the marginal rate of substitution, normality of both goods. Subtracting $(I - 1)t$,

$$p(2l) \geqslant \beta - \frac{f}{4(l - I)} \geqslant \frac{f(17/16)}{I} - \frac{f}{4(l - I)} \geqslant \frac{f(I + 1)}{2I^2} - \frac{f}{4(l - I)}$$
$$\geqslant \frac{fl}{2I(l - I)} - \frac{f}{4(l - I)}$$

by assumption on f/β, I and l, respectively. This contradicts the presumption and thus completes the proof of the nonnegativity of $p(x)$.

Equation (3) is verified by substitution of the expressions above for consumption and output (note that the transportation cost terms cancel). Equations (4)–(7) hold by construction.

Next, we argue that the allocation we have specified solves the consumers' problems (1). By construction of the exchange economy equilibrium, no consumer has an incentive to relocate within the intervals occupied by the consumers. The land occupied by producers is less expensive than any land occupied by consumers, but always requires more transport cost. Consider a consumer parcel $(a, a + s)$ containing part of the land parcel of the firm located at $(l - I, l + I)$. We may assume that $a + s/2 \leqslant l$. For if $a + s/2 > l$, then we can flip the consumer parcel symmetrically about l, save on commuting cost, and obtain the same quantity of land.

First we consider the case $a + s > l + I$. The idea is to shift the parcel towards the left. This saves commuting cost. It also saves rent, as long as $p(a) \leqslant p(a + s)$. By symmetry about l, rent density $p(a + s)$ is also attained at $2l - (a + s)$, but a is to the left of this point, since $a + s/2 \leqslant l$. The next point leftward where rent density $p(a + s)$ is attained is $-2l + (a + s)$, by symmetry about 0. As long as $s \leqslant 2l$, a is to the right of $-2l + (a + s)$ and we can shift the parcel towards the left, saving both commuting cost and rent. If $s > 2l$, then since $a + s/2 \leqslant l$, $a < 0$; now we will show that the utility associated with such a big parcel is below the equilibrium utility level of consumers. We distinguish two sub-cases. Call the rightmost consumer commuting to the left producer consumer i. In the first sub-case, $a \leqslant a_i$. The encroaching consumer is spending at least as much on land as any consumer in equilibrium, is consuming at least as much land, and is facing the same marginal commuting cost. Therefore, using strict quasi-concavity, the marginal willingness to pay of this encroaching consumer for land to the left of a_i is no more than the marginal willingness to pay of consumer i. So parcels containing points to the left of a_i will yield lower utility. Now consider the second sub-case, $a_i < a < 0$. By shifting the parcel to the left, towards the left producer, the quantity of land consumed is the same, and the

savings in commuting cost (t per unit distance) exceed the additional rent, $p(a) - p(a+s)$. This inequality follows from three facts. First, since we are in the declining rent region, $p(a) < p(a_i)$. Second, $p(a+s) \geqslant p(0)$, the minimum consumer rent density (recall that $a + s > l + I$, so $a + s$ is in a consumer's parcel). Third, $p(a_i) - p(0) = t$, the first-order condition of consumer i with respect to a. Thus, a shift to the left increases utility and we conclude that it suffices to consider $a + s \leqslant l + I$.

Summarizing, ruling out $a \leqslant a_i$ as before, and using the fact that very small consumer parcels will only be located on the left part of the firm's parcel, $(l - I, l + I)$, to save commuting cost, the only choices that might be optimizing and yielding higher utility than equilibrium utility for any consumer are:

for $s < 2I$ (the size of the firm's parcel), $(l - I, l - I + s)$,

for $2I \leqslant s \leqslant l + I + s_i$ (or $a_i \leqslant a \leqslant l - I$), $(a, l + I)$.

In the first case, by assumption, $l \geqslant 2I^2 + I$, $s < 2I \leqslant (l - I)/I \leqslant s_i$. If the encroaching consumer has a greater utility level than consumer i, then we reduce his composite good consumption until the utility levels are the same. By strict quasi-concavity, the marginal willingness to pay for land is greater for the encroaching consumer. By the first order conditions the rent density he faces on the right hand side of his parcel must exceed that of consumer i. This contradicts the construction of the rent schedule.

In the second case the parcel is $(a, l + I)$. If $a > 0$, let us compare this parcel to an alternative parcel, $(a - 2I, l - I)$, that is the same size but just does not encroach on the producer. Since $a > 0$ and the alternative parcel does not encroach, the consumer saves at least $(l - I)t$ in commuting cost by moving to the alternative, which is adjacent to a producer. An upper bound on the additional cost of land is the difference between the maximal and minimal prices of land over a parcel of size $2I$, $2I(I - 1)t + f(l - I/2)/(l - I)$. This is less than $(l - I)t$, by the assumptions on f and t (yielding $f/t \leqslant (16/9)I$) and on l (the lower bound is a worst case) and I. Summarizing, the alternative parcel (that does not encroach on a producer), $(a - 2I, l - I)$, is the same size as the original parcel, $(a, l + I)$, and after paying for commuting cost, there is at least as much consumption good remaining. Thus, the only parcel choices that might be optimizing and yielding higher utility than equilibrium utility are $(a, l + I)$ where $a_i \leqslant a \leqslant 0$.

If $a_i \leqslant a \leqslant 0$, then the amount of land purchased exceeds $l - I$, hence s_i, and therefore the marginal willingness to pay for land is less than $p(2l)$. Hence the consumer must therefore be willing to purchase more land, beyond the point 0, only if

$$\int_0^{l-I} p(x)\,\mathrm{d}m(x) + \int_{l-I}^{l+I} p(x)\,\mathrm{d}m(x) \leqslant \int_0^{l-I} p(2l)\,\mathrm{d}m(x) + \int_{l-I}^{l+I} p(2l)\,\mathrm{d}m(x),$$

or

$$\int_0^{l-I} \big[p(x) - p(2l)\big]\,\mathrm{d}m(x) \leqslant 2If \cdot \frac{l/(2I) - 1/4}{l - I}. \tag{13}$$

Next, we contradict this inequality by using our assumptions. In the proof of the non-negativity of the firms's rent the combination $c_1 \geqslant \bar{c}$ and $s_1 \leqslant \bar{s}$ was shown to contradict

the assumption on the marginal rate of substitution. Two possibilities remain: $s_1 > \bar{s}$ or $c_1 < \bar{c}$. If $s_1 > \bar{s}$, then $s_1 > 4f[l/(2I) - 1/4]/[(l + I)(I - 1)t]$, so $s_1(I - 1)t > 4f[l/(2I) - 1/4]/(l + I)$. Now $s_2(I - 2)t \geqslant s_1(I - 2)t, \ldots, s_{I-1}t \geqslant s_1 t$. Summing these inequalities and using $1 + 2 + \cdots + I - 1 = (I - 1)I/2$, we obtain

$$\int\limits_0^{l-I} \left[p(x) - p(2l) \right] \mathrm{d}m(x) > \frac{\bar{s}(I - 1)It}{2} = 2If \cdot \frac{l/(2I) - 1/4}{l + I},$$

contradicting inequality (13).

Now consider the remaining case, $c_1 < \bar{c}$ and $s_1 < \bar{s}$. Use the lower bounds for transport cost and rent on $[0, l - I)$: $ts_1 + \cdots + t(I - 1)s_1 = t(I - 1)Is_1/2$ and $p(2l)(l - I) + (I - 1)ts_1 + \cdots + ts_1 = p(2l)(l - I) + (I - 1)Its_1/2$, respectively. Then using c_1 (the consumption of the first consumer) as a lower bound on the consumption on the interval $[0, l - I)$, $c_1 + p(2l)(l - I) + (I - 1)Its_1 \leqslant Iw^* = Ic_1 + I[p(2l) + (I - 1)t]s_1$. Hence, using the non-negativity of the firms rent, $c_1 \geqslant [p(2l)(l - I) - Ip(2l)s_1]/(I - 1) \geqslant f[l/(2I) - 1/4](l - I - I\bar{s})/[(l + I)(I - 1)] \geqslant \bar{c}$ by definition of $\bar{c}$, contradicting $c_1 < \bar{c}$.

Thus, when transport costs are taken into account, the willingness to pay of a consumer for any land occupied by a producer falls short of the cost. A consumer purchasing land used by a producer will have utility lower than a consumer farthest away from a producer. Since all consumers are at the same utility level in equilibrium, such a purchase would reduce the utility level of the consumer, and therefore will not be made.

With regard to the firms, notice that optimization will imply that the labor input quantity will be set equal to the land input quantity, and optimization problem (2) reduces to:

$$\max_{b,\sigma} \beta \cdot \sigma - f - \int\limits_b^{b+\sigma} p(x)\,\mathrm{d}m(x) - w^* \cdot \sigma.$$

The first-order condition with respect to σ is $\beta - w^* = p(b + \sigma) \in [p(l), p(l + I)]$. Marginal revenue net of labor cost equals the marginal cost of land. Since there is a discontinuity in the price of land, this net marginal revenue need only be between the bounds of the discontinuity. w^* was chosen to satisfy this condition for $b_1 = l - I, \sigma_1 = 2I$, $b_2 = -l - I, \sigma_2 = 2I$. The first-order condition with respect to b is $p(b) = p(b + \sigma)$; this is fulfilled by symmetry. Equilibrium profits are zero by construction of w^*; see Eq. (12).

Turning next to second-order conditions for the firm, notice first that if the firm uses a parcel of any size σ, it is indifferent about its location, so it will choose one of the cheapest parcels. For $\sigma \leqslant 2I$, these are contained in $(b_1, b_1 + \sigma_1)$, $(b_2, b_2 + \sigma_2)$. The first-order condition with respect to σ will imply that it will choose $\sigma = 2I$. If it occupies a parcel at an extreme of X and σ is slightly larger than $2I$, then the cost of this parcel is higher than the cost of a similarly slight extension of $(b_1, b_1 + \sigma_1)$ or $(b_2, b_2 + \sigma_2)$. If the firm can make higher profits from expanding the scale of its operations beyond $2I$, then given the production function and the price density, it will make still higher profits when $b = -2l$ and $\sigma = 4l$.

Profits from such a production plan are given by

$$4\beta l - f - w^* \cdot 4l - \int_{-2l}^{2l} p(x)\,\mathrm{d}m(x). \tag{14}$$

Profits from the equilibrium production plan are zero by construction of w^*. Using this by substituting the definition of w^* given by Eq. (12) into Eq. (14), after some tedious calculations, non-positivity of (14) is equivalent to $\int_0^{l-I}[p(x) - p(2l)]\,\mathrm{d}m(x) \geqslant 0$. The integrand is non-negative by construction. $\square$

Proof of Theorem 4. Take an equilibrium allocation

$$\left[(c_i, a_i, s_i, Q_i)_{i=1}^I, c_{\mathrm{L}}, (z_j, b_j, \sigma_j, q_j)_{j=1}^J\right]$$

that is symmetric in production, and suppose that it is Pareto dominated by another feasible allocation

$$\left[(c_i', a_i', s_i', Q_i')_{i=1}^I, c_{\mathrm{L}}', (z_j', b_j', \sigma_j', q_j')_{j=1}^J\right],$$

with $z_j' > 0 \; \forall j$. So $u(c_i', s_i') \geqslant u(c_i, s_i)$ for all i and $c_{\mathrm{L}}' \geqslant c_{\mathrm{L}}$, with strict inequality holding for at least one relation.

First,[18] we assert that without loss of generality, we can assume that the land parcels of consumers commuting to a firm in the Pareto dominating allocation form a connected set in combination with that firm's parcel. For if not, we can switch the land parcels around so that they do form a connected set, and create a Pareto improvement by reducing aggregate commuting cost and distributing the surplus composite commodity to the landlord.

Second, we argue that without loss of generality, the Pareto dominating allocation has the same number of consumers commuting to each firm from each side or direction. By the first condition defining an allocation that is symmetric in production, $I/(2J)$ is integer. All consumers commute (see footnote 6). It follows that the difference between the maximum and minimum number of consumers commuting to any firm from one side at the Pareto dominating allocation, $\bar{n}$ and $\underline{n}$ respectively, must be more than one. (The proof is by contradiction. There are $2J$ clusters of consumers (to the left and to the right of the J firms). Let the number of clusters with $\underline{n}$ consumers be N, $0 < N < 2J$. Now suppose $\bar{n} = \underline{n} + 1$. Then $I = \underline{n}N + (\underline{n}+1)(2J - N) = (\underline{n}+1)2J - N$. Dividing by $2J$ we obtain that $N/(2J)$ is integer, contradicting $0 < N < 2J$.) Take the closest consumer, consumer 1, commuting to a firm from a side with $\bar{n}$ consumers commuting to the firm. Move this consumer, retaining their land and composite good consumption, to the side of a firm with $\underline{n}$ consumers commuting to it. Place this consumer so that it is the agent adjacent to the firm on the side with $\underline{n}$ consumers commuting to it. Shift agents (without changing their order) so that material balance is maintained in the land market.

We claim that this rearrangement of consumers creates a Pareto improvement. The reason is as follows. Removing the first consumer from the side with $\bar{n}$ consumers reduces

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[18] At this juncture, it is important to note that the concept of "Pareto optimality with J active firms" implies that no firm is shut down in the Pareto dominating allocation.

total commuting cost from that side by $(\bar{n} - 1) \cdot s_1' \cdot t$. Placing the consumer in the side with $\underline{n}$ commuters increases commuting cost by $\underline{n} \cdot s_1' \cdot t$, where $\underline{n} < \bar{n} - 1$. Thus, a surplus of composite good is created, and this can be given to the landlord.

Since $I/(2J)$ is integer, it must be that each firm has the same number of consumers commuting to it from each side. From the form of the production function, we know that the production plans of all firms must therefore be identical, since labor usage is identical (and equal to I/J).

Third, we claim that without loss of generality, the Pareto improving allocation has the property that the consumers adjacent to a firm all have the same allocations of consumption good and land, the consumers second closest to a firm all have the same allocations, and so forth. For suppose that this were not the case. Take the set of all of the consumers who are i people from the firm to which they are commuting. Take the average of their allocations and give each of them the average allocation. Do this separately for each set of consumers who are i people from each firm. This new, average allocation is feasible since the original allocation is feasible. For instance, aggregate commuting cost is the same in both the original and averaged allocations. Moreover, since utility is strictly quasi-concave, the original allocation Pareto dominates the equilibrium allocation, and the equilibrium allocation features equal utility levels for all consumers (see footnote 6), the average allocation also Pareto dominates the equilibrium allocation.

An immediate implication is that the Pareto dominating allocation is, without loss of generality, symmetric in production. Since the equilibrium allocation is symmetric in production (by assumption), the locations of producers and their land usage are the same in both the equilibrium allocation and the Pareto dominating allocation.[19] Thus, the difference boils down to a pure exchange economy where the central business districts are the firms and the consumers are each endowed with w units of consumption good. From Berliant and Fujita [9, Proposition 2], given a fixed production sector, the equilibrium allocation is efficient. This contradicts the presumed existence of a Pareto dominating allocation. So the hypothesis is false, and the equilibrium allocation is Pareto optimal. $\quad\square$

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