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Bias and Sensitivity of Multipliers

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ABSTRACT *Multipliers measure the derivatives of endogenous variables with respect to exogenous shocks and are functions of the structural parameters of an economic model. Substitution of the structural parameter estimates yields a so-called derived estimate for a multiplier or any reduced-form parameter. Derived estimates are biased. This paper presents first-order approximations to the biases and sensitivities of multipliers. The good performance of a flawed formula in input–output analysis is illuminated.*

KEYWORDS: *Multipliers, bias, input–output analysis*

1. Introduction

For non-linear functions, the mean value differs from the value of the mean. However, it is standard to estimate derived constructs, such as multipliers, simply by evaluating the functions that define the constructs at the estimated values of the model coefficients. For example, in input–output (IO) analysis, multipliers are estimated by taking the Leontief inverse of the estimated IO coefficients. This procedure is biased and the bias is equal to the difference between the mean of the function value and the function value of the mean. In this paper, we analyze the bias and sensitivity of multipliers from the general perspective of functions of random variables. Simple first-order approximations are established and then applied to multiplier analysis.

In Section 2, we consider functions of random variables, $\mathbf{b} = \boldsymbol{\beta}(\mathbf{a})$, and formally define the bias and variance of the estimator $\boldsymbol{\beta}(\hat{\mathbf{a}})$, where $\hat{\mathbf{a}}$ is the estimate of $\mathbf{a}$. Here, $\mathbf{a}$ is a structural variable and $\boldsymbol{\beta}(\mathbf{a})$ is a reduced-form parameter. We are interested in the transmission of errors. First-order approximations to the bias and variance of $\boldsymbol{\beta}(\mathbf{a})$ are given. In Section 3, we relate our work to an alternative statistical methodology, which prevails in IO analysis.

In economics, more generally, one is often interested in the dependence of endogenous variables on exogenous variables, but the relationship is given only implicitly, by means of a model. The dependence is given by the implicit function

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theorem, which relates the derivatives of the endogenous variables with respect to the exogenous variables to the model coefficients. The relationship is non-linear, as the implicit function theorem involves the quotient of two matrices. Section 4 formalizes these concepts, including multipliers, and assesses the bias and sensitivity of their standard estimates, by simple application of the first-order approximations of Section 2. An economic model $\mathbf{f}$ relates exogenous variables, listed in $\mathbf{x}$, and endogenous variables, listed in $\mathbf{y}$, through a system of equations $\mathbf{f}(\mathbf{x},\mathbf{y}) = 0$, and the multipliers are $\partial \mathbf{y}/\partial \mathbf{x}$. A simple example is where $\mathbf{f}$ has components $-\mathbf{I}$ and $\mathbf{I}-\mathbf{A}$, constituting input–output (IO) analysis. Section 5 specializes the results of Section 4 to this case, obtaining easy bias and sensitivity formulae. Section 6 concludes.

2. Bias and Sensitivity Approximations

When an estimated model is employed for the evaluation of functions of parameters, we are in the business of 'derived estimation'. For example, if a denotes the propensity to consume, then the Keynesian multiplier is $b = \beta(a) = (1-a)^{-1}$. The most common derived estimator is the value of the function at the model estimate: if a is estimated by $\hat{a}$, then b is estimated by $\beta(\hat{a})$. The direct model estimate $\hat{a}$ is employed to obtain a derived estimate of some construct of concern. The notation $\hat{a}$ reflects the random nature of estimates.

In general, the estimated model coefficients are listed in a vector of random variables $\hat{\mathbf{a}}$, with mean $E(\hat{\mathbf{a}}) = \mathbf{a}$ and variance–covariance matrix $V(\hat{\mathbf{a}})$. Elements of $V(\hat{\mathbf{a}})$ will be denoted by $v(\hat{a}_i, \hat{a}_j)$. A simultaneous system of equations estimate, given by coefficients matrix $\hat{\mathbf{A}}$, can be cast in this framework, by stacking the elements in a vector. The derived constructs (multipliers, elasticities, etc.) are listed in a vector of random variables, i.e.

$$\hat{\mathbf{b}} = \boldsymbol{\beta}(\hat{\mathbf{a}}) \tag{1}$$

where the function $\boldsymbol{\beta}$ maps vectors to vectors (of possibly different dimension). For non-linear functions $\boldsymbol{\beta}$, the derived estimator $\hat{\mathbf{b}}$ has a bias B given by

$$B(\hat{\mathbf{b}}) \approx E(\hat{\mathbf{b}}) - \mathbf{b} \tag{2}$$

and overestimates $\mathbf{b} = \boldsymbol{\beta}(\mathbf{a})$ by this quantity. The purpose of the analysis is to approximate the mean $E(\hat{\mathbf{b}})$ and the variance $V(\hat{\mathbf{b}})$ of the derived constructs $\hat{\mathbf{b}}$. To this end, define matrices $\boldsymbol{\beta}'(\mathbf{a})$ of first derivatives and $\boldsymbol{\beta}_j''(\mathbf{a})$ of second derivates in $\mathbf{a} = E(\hat{\mathbf{a}})$ by

$$\beta'(\mathbf{a})_{pq} \approx \frac{\partial \beta_q}{\partial \hat{a}_p}(\mathbf{a}), \qquad \beta_j''(\mathbf{a})_{pq} \approx \frac{\partial^2 \beta_j}{\partial \hat{a}_p \partial \hat{a}_q}(\mathbf{a}) \tag{3}$$

The sensitivity of the direct estimate is given by its variance. The variance of a derived estimator has a well-known approximation (Klein, 1956; Kmenta, 1971), i.e.

$$V(\hat{\mathbf{b}}) \approx \boldsymbol{\beta}'(\mathbf{a})^{\mathrm{T}} V(\hat{\mathbf{a}}) \boldsymbol{\beta}'(\mathbf{a}) \tag{4}$$

which can be derived using a first-order Taylor expansion. The formula is exact for linear functions and holds approximately for small variances, such that $\boldsymbol{\beta}(\hat{\mathbf{a}})$ can be linearized about $\boldsymbol{\beta}(\mathbf{a})$. In other words, the direct model estimate must be precise relative to the local behaviour of the function, in order to make inferences about the value of the derived construct. This requirement is reasonable and will be maintained throughout the paper.

Although the bias is a lower-order moment than is the variance, it can be approximated by using a second-order Taylor expansion. Defining the linear increment $\Delta\hat{\mathbf{a}} \approx \hat{\mathbf{a}} - \mathbf{a}$, this expansion reads

$$\hat{b}_j - b_j \approx [\boldsymbol{\beta}'(\mathbf{a})^{\mathrm{T}}\Delta\hat{\mathbf{a}}]_j + \frac{1}{2}\Delta\hat{\mathbf{a}}^{\mathrm{T}}\boldsymbol{\beta}_j''(\mathbf{a})\Delta\hat{\mathbf{a}} \tag{5}$$

with the remaining terms being of higher order in $\Delta\hat{\mathbf{a}}$. Taking expectations, the left-hand side becomes the jth component of the bias defined in equation (2) and the first term on the right-hand side vanishes by definitions of $\Delta\hat{\mathbf{a}}$ and mean a. Consequently, we have

$$B(\hat{b})_j \approx \frac{1}{2}\boldsymbol{\beta}_j''(\mathbf{a}) \cdot V(\hat{\mathbf{a}}) \tag{6}$$

plus terms of higher order in $V(\hat{\mathbf{a}})$. Here, $\cdot$ denotes the inner product, which is extended to matrices by stacking the elements in vectors. The approximation to the bias is exact if and only if $\boldsymbol{\beta}$ is quadratic. The approximation is positive when $\boldsymbol{\beta}$ is convex at $\mathbf{a}$ and negative when $\boldsymbol{\beta}$ is concave at $\mathbf{a}$. This is in agreement with Jensen's inequality. In fact, the derived bias approximation quantifies the gap in Jensen's inequality. Not surprisingly, the second derivative is crucial.

3. Methodology

An alternative strand of literature (Jackson & West, 1989, p. 213) speaks of underestimation when quantity (2) is positive. Those with this perspective, position themselves differently in two respects. First, the subject of analysis is the mean of the construct of interest, rather than some underlying true value. Secondly, this model is considered to be known by its unbiased estimate. From this point of view, parameters are intrinsically random, without underlying 'true' values. This perspective is Bayesian in spirit. By the second aspect, $\hat{\mathbf{a}}$ is essentially equated with $\mathbf{a}$ in the further, derived estimation. The value of the function $\boldsymbol{\beta}$ at this point is $\boldsymbol{\beta}(\mathbf{a}) = \mathbf{b}$ and underestimates $E(\hat{\mathbf{b}})$, i.e. the subject of interest. The above bias must be added rather than subtracted. It is important to disentangle the differences, since the literature is confused. Jackson and West (1989, p. 213) write that a misinterpretation by Lahiri and Satchell (1986) of Simonovits' (1975) result led to a badly needed, but largely unsuccessful, attempt to unravel terminology. They ascribe the misinterpretation to the interchangeable usage of the terms 'true' and 'expected'.

The first point of difference is the specification of the relevant subject of analysis: $\mathbf{b}$ versus $E(\hat{\mathbf{b}})$. This is not an analytical issue and it can be settled independently of the other difference. The choice between $\mathbf{b}$ and $E(\hat{\mathbf{b}})$ is a choice between the value of the function at the mean and the mean of the value of the function. The function value at the mean is appropriate when the derived construct is of interest in itself; for example, the notion of a total propensity to consume, $b = (1 - a)^{-1}$, where a is the Keynesian propensity to consume. The mean of the function value is relevant when the derived construct is a vehicle to assess economic effects, such as the multiplier effect of a Keynesian expenditure programme, $E(\hat{b}) = E[(1 - \hat{a})^{-1}]$. If this is the subject of interest, then $\hat{b}$ is an unbiased estimator and no correction needs be made, explaining one-half of the reversal from overestimation to underestimation. To focus on the second point of difference, fix the subject of estimation on $E(\hat{\mathbf{b}})$, for instance.

An example illuminates. Consider $b = \beta(a) = a^2$. Let $\hat{a}$ be an estimate of a: $\hat{a} = a + \varepsilon$. Then, $\hat{b} = a^2 + 2\varepsilon a + \varepsilon^2$. Because of the last term, the mean of $\hat{b}$ exceeds $a^2 = b$. If we know $a = E(\hat{a})$, then we must augment $\beta(a)$ to estimate the mean of b. This presumption makes sense when the model is considered to be purely random and estimated. If we do not know a, then we take $\beta(\hat{a})$ without correction and the inclusion of ε^2 is automatic.

4. Multiplier Analysis

An economic model $\mathbf{f}$ relates exogenous variables, listed in $\mathbf{x}$, and endogenous variables, listed in $\mathbf{y}$, through a system of equations

$$\mathbf{f}(\mathbf{x}, \mathbf{y}) = 0 \tag{7}$$

When the equations are specified by principles of economic theory, they constitute a structural form model. In scenario analysis, the effects of exogenous shocks on the endogenous variable are traced. These effects are essentially reduced-form coefficients $\mathbf{b}$, which can be obtained by the implicit function theorem

$$\mathbf{b} = \frac{\partial \mathbf{y}}{\partial \mathbf{x}} = -\mathbf{f}'_2(\mathbf{x}, \mathbf{y})^{-1}\mathbf{f}'_1(\mathbf{x}, \mathbf{y}) = \beta[\mathbf{f}'(\mathbf{x}, \mathbf{y})] \tag{8}$$

Here, $\mathbf{A} = \mathbf{f}'_2(\mathbf{x}, \mathbf{y})$ and $\mathbf{C} = \mathbf{f}'_1(\mathbf{x}, \mathbf{y})$ comprise the structural form coefficients of the endogenous and exogenous variables $\mathbf{y}$ and $\mathbf{x}$ respectively. $\boldsymbol{\beta}$ is the matrix-valued function defined by

$$\boldsymbol{\beta}\begin{pmatrix}\mathbf{C}\\ \mathbf{A}\end{pmatrix} = -\mathbf{A}^{-1}\mathbf{C} \tag{9}$$

The structural form model is considered to be estimated by $\hat{\mathbf{f}}$. Exogenous variables remain deterministic, but endogenous variables become random, as a result of the imprecision of the model estimate

$$\hat{\mathbf{f}}(\mathbf{x}, \hat{\mathbf{y}}) = \mathbf{0} \tag{10}$$

The deterministic nature of exogenous variables permits maintenance of the multiplier analysis, yielding derived estimates that are now collected in a matrix

$$\frac{\partial \hat{\mathbf{y}}}{\partial \mathbf{x}} = \boldsymbol{\beta}[\hat{\mathbf{f}}'(\mathbf{x}, \hat{\mathbf{y}})] = -\hat{\mathbf{A}}^{-1}\hat{\mathbf{C}} \tag{11}$$

with $\hat{\mathbf{A}} = \hat{\mathbf{f}}'_2(\mathbf{x}, \hat{\mathbf{y}})$ and $\hat{\mathbf{C}} = \hat{\mathbf{f}}'_1(\mathbf{x}, \hat{\mathbf{y}})$. These multiplier or reduced-form estimates depend on the direct model or structural form estimates

$$\hat{\mathbf{a}} = \begin{pmatrix}\hat{\mathbf{C}}\\ \hat{\mathbf{A}}\end{pmatrix} \quad \text{(vectorized)}$$

through the function $\boldsymbol{\beta}$. Since $\boldsymbol{\beta}$ is non-linear, the multiplier estimates are biased. In equation (5), the jth component of the bias was approximated by half the mean of $\Delta\hat{\mathbf{a}}^{\mathrm{T}}\boldsymbol{\beta}_j''(a)\Delta\hat{\mathbf{a}}$. This is a quadratic form in $\Delta\hat{\mathbf{a}}$, so can be found by differentials about $\begin{pmatrix}\mathbf{C}\\ \mathbf{A}\end{pmatrix}$.

Rewrite equation (11) as

$$\hat{\mathbf{A}}\frac{\partial\hat{\mathbf{y}}}{\partial\mathbf{x}} = -\hat{\mathbf{C}}$$

and differentiate, to find

$$\hat{\mathbf{A}}\,\mathrm{d}\frac{\partial\hat{\mathbf{y}}}{\partial\mathbf{x}} + (\mathrm{d}\hat{\mathbf{A}})\frac{\partial\hat{\mathbf{y}}}{\partial\mathbf{x}} = -\mathrm{d}\hat{\mathbf{C}} \tag{12}$$

or, introducing $\hat{\mathbf{Z}} = \hat{\mathbf{A}}^{-1}$, we have

$$\mathrm{d}\frac{\partial\hat{\mathbf{y}}}{\partial\mathbf{x}} = -\hat{\mathbf{Z}}(\mathrm{d}\hat{\mathbf{A}})\frac{\partial\hat{\mathbf{y}}}{\partial\mathbf{x}} - \hat{\mathbf{Z}}\,\mathrm{d}\hat{\mathbf{C}} \tag{13}$$

Differentiating once more yields

$$\mathrm{d}^2\frac{\partial\hat{\mathbf{y}}}{\partial\mathbf{x}} = -(\mathrm{d}\hat{\mathbf{Z}})(\mathrm{d}\hat{\mathbf{A}})\frac{\partial\hat{\mathbf{y}}}{\partial\mathbf{x}} - \hat{\mathbf{Z}}(\mathrm{d}\hat{\mathbf{A}})\,\mathrm{d}\frac{\partial\hat{\mathbf{y}}}{\partial\mathbf{x}} - (\mathrm{d}\hat{\mathbf{Z}})(\mathrm{d}\hat{\mathbf{C}}) \tag{14}$$

Now, substitute $\mathrm{d}\hat{\mathbf{Z}} = -\hat{\mathbf{Z}}(\mathrm{d}\hat{\mathbf{A}})\hat{\mathbf{Z}}$ (consequence of $\hat{\mathbf{A}}\hat{\mathbf{Z}} = \mathbf{I}$) and equation (13) for $\mathrm{d}(\partial\hat{\mathbf{y}}/\partial\mathbf{x})$, evaluate at $\begin{pmatrix}\mathbf{C}\\ \mathbf{A}\end{pmatrix}$, $\mathbf{Z} = \mathbf{A}^{-1}$ and $\partial\mathbf{y}/\partial\mathbf{x} = -\mathbf{ZC}$, to find

$$\mathrm{d}^2\frac{\partial\hat{\mathbf{y}}}{\partial\mathbf{x}} = -2\mathbf{Z}(\mathrm{d}\hat{\mathbf{A}})\mathbf{Z}(\mathrm{d}\hat{\mathbf{A}})\mathbf{ZC} + 2\mathbf{Z}(\mathrm{d}\hat{\mathbf{A}})\mathbf{Z}(\mathrm{d}\hat{\mathbf{C}}) \tag{15}$$

It follows that

$$\Delta^2\frac{\partial\hat{\mathbf{y}}}{\partial\mathbf{x}} \approx -2\mathbf{Z}(\Delta\hat{\mathbf{A}})\mathbf{Z}(\Delta\hat{\mathbf{A}})\mathbf{ZC} + 2\mathbf{Z}(\Delta\hat{\mathbf{A}})\mathbf{Z}(\Delta\hat{\mathbf{C}}) \tag{16}$$

Therefore, taking half the expectation in equation (16), we have that

$$B\left(\frac{\partial\hat{\mathbf{y}}}{\partial\mathbf{x}}\right) \approx -E[\mathbf{Z}(\Delta\hat{\mathbf{A}})\mathbf{Z}(\Delta\hat{\mathbf{A}})\mathbf{ZC}] + E[\mathbf{Z}(\Delta\hat{\mathbf{A}})\mathbf{Z}(\Delta\hat{\mathbf{C}})] \tag{17}$$

Multiplier analysis makes sense only if the structural coefficients are known quite precisely relative to the local changes considered. It is in the sense that the approximation is of the first order. The variances of the structural coefficients are assumed to be small. This is a strong assumption, which frees the analysis from distributional specifications other than the means and variances.

$(\partial\hat{\mathbf{y}}/\partial\mathbf{x})_{hk}$ estimates the multiplier of endogenous variable h with respect to exogenous variable k. Taking this component, we have

$$B\left(\frac{\partial\hat{y}_h}{\partial x_k}\right) \approx -\sum_{i,j,l,m,n} z_{hi}z_{jl}z_{mn}c_{nk}v(\hat{a}_{ij}, \hat{a}_{lm}) + \sum_{i,j,l} z_{hi}z_{jl}v(\hat{a}_{ij}, \hat{c}_{lk}) \tag{18}$$

Recall that, in this formula, $\mathbf{Z}=\mathbf{A}^{-1}$. An alternative derivation is by evaluation of $\beta''\begin{pmatrix}\mathbf{C}\\ \mathbf{A}\end{pmatrix}$. (In fact, we have

$$\frac{\partial^2\beta_{hk}}{\partial a_{ij}\,\partial a_{lm}}=-2z_{hi}z_{jl}\Sigma_n z_{mn}c_{nk}$$

while

$$\frac{\partial^2\beta_{hk}}{\partial a_{ij}\,\partial c_{lm}}=\begin{cases}z_{hi}z_{jl}, & k=m\\ 0, & \text{otherwise}\end{cases}$$

Keep in mind that the cross-partials enter

$$\frac{1}{2}\beta''\begin{pmatrix}\mathbf{C}\\ \mathbf{A}\end{pmatrix}\cdot V(\hat{\mathbf{a}})$$

twice.) The variances of the endogenous variables coefficients and the covariances with the exogenous variables coefficients enter the bias expression, but the variances of the exogenous variables coefficients do not. The multipliers are linear in the exogenous variables coefficients, so are not biased by those. This completes the bias analysis. The sensitivity analysis is analogous, as we will see now.

Evaluate the first-order differential of the multipliers at $\begin{pmatrix}\mathbf{C}\\ \mathbf{A}\end{pmatrix}$, $\mathbf{Z}=\mathbf{A}^{-1}$ and $\partial\mathbf{y}/\partial\mathbf{x}=-\mathbf{ZC}$; then, we have

$$\mathrm{d}\frac{\partial\hat{\mathbf{y}}}{\partial\mathbf{x}}=\mathbf{Z}(\mathrm{d}\hat{\mathbf{A}})\mathbf{ZC}-\mathbf{Z}(\mathrm{d}\hat{\mathbf{C}}) \tag{19}$$

It follows that the linear form is

$$\Delta\frac{\partial\hat{\mathbf{y}}}{\partial\mathbf{x}}=\mathbf{Z}(\Delta\hat{\mathbf{A}})\mathbf{ZC}-\mathbf{Z}(\Delta\hat{\mathbf{C}}) \tag{20}$$

and, squaring and taking the expectation, we have

$$V\left(\frac{\partial\hat{\mathbf{y}}}{\partial\mathbf{x}}\right)\approx E([\mathbf{Z}(\Delta\hat{\mathbf{A}})\mathbf{ZC}-\mathbf{Z}(\Delta\hat{\mathbf{C}})]^{\mathrm{T}}[\mathbf{Z}(\Delta\hat{\mathbf{A}})\mathbf{ZC}-\mathbf{Z}(\Delta\hat{\mathbf{C}})]) \tag{21}$$

Taking the components (h,k) and (r,s), we find

$$\begin{aligned}v\left(\frac{\partial\hat{y}_h}{\partial x_k},\frac{\partial\hat{y}_r}{\partial x_s}\right)\approx&\sum_{i,j,n,l,m,p} z_{hi}z_{jn}c_{nk}z_{rl}z_{mp}c_{ps}v(\hat{a}_{ij},\hat{a}_{lm})\\ &+\sum_{i,l} z_{hi}z_{rl}v(\hat{c}_{ik},\hat{c}_{ls})\\ &+\sum_{i,j,n,l} z_{hi}z_{jn}c_{nk}z_{rl}v(\hat{a}_{ij},\hat{c}_{ls})\\ &+\sum_{i,l,m,p} z_{hi}z_{rl}z_{mp}c_{ps}v(\hat{c}_{ik},\hat{a}_{lm})\end{aligned} \tag{22}$$

These bias and variance formulae can be evaluated on the basis of the direct estimates, without inversion.

5. Application to IO Analysis

The two papers that best represent the state of stochastic IO research are those of Lahiri and Satchell (1986) and West (1986), according to Jackson and West (1989). While Lahiri and Satchell's (1986) shortcomings are methodological (as discussed in Section 3), West (1986) is flawed from a mathematical point of view. (See also Kop Jansen (1994) for an up-to-date review of the field.) In the present paper, we pursue our general first-order analysis.

IO analysis is a special case of multiplier analysis. The exogenous variables ($\mathbf{x}$) are final demand and the endogenous variables ($\hat{\mathbf{y}}$) are total output. (The notation is perverse from the IO point of view, but will be maintained.) The variables are related by a matrix of technical coefficients $\hat{\mathbf{A}}$, such that

$$(\mathbf{I}-\hat{\mathbf{A}})\hat{\mathbf{y}}=\mathbf{x} \tag{23}$$

In other words, the economic model of Section 4 is

$$\hat{\mathbf{f}}\begin{pmatrix}\mathbf{x}\\ \hat{\mathbf{y}}\end{pmatrix}=(-\mathbf{I}\ \ \mathbf{I}-\hat{\mathbf{A}})\begin{pmatrix}\mathbf{x}\\ \hat{\mathbf{y}}\end{pmatrix} \tag{24}$$

with $\hat{\mathbf{f}}_2'=\mathbf{I}-\hat{\mathbf{A}}$ and $\hat{\mathbf{f}}_1'=\mathbf{C}=-\mathbf{I}$. Consequently, the multipliers are given by

$$\frac{\partial\hat{\mathbf{y}}}{\partial\mathbf{x}}=\beta(\hat{\mathbf{f}}')=-(\hat{\mathbf{f}}_2')^{-1}\hat{\mathbf{f}}_2'=(\mathbf{I}-\hat{\mathbf{A}})^{-1}=\hat{\mathbf{B}} \tag{25}$$

which is the Leontief inverse of $\hat{\mathbf{A}}$. Since $(\mathbf{I}-\hat{\mathbf{A}})$ has the same variance-covariance matrix as $\hat{\mathbf{A}}$ and all the covariances with $\hat{\mathbf{C}}$ vanish, equations (18) and (22) apply with $\mathbf{C}=-\mathbf{I}$. With $\hat{\mathbf{B}}=(\mathbf{I}-\hat{\mathbf{A}})^{-1}$, the formulae become

$$B(\hat{b}_{hk})\approx\sum_{i,j,l,m}b_{hi}b_{jl}b_{mk}v(\hat{a}_{ij},\hat{a}_{lm}) \tag{26}$$

and

$$v(\hat{b}_{hk},\hat{b}_{rs})\approx\sum_{i,j,l,m}b_{hi}b_{jk}b_{rl}b_{ms}v(\hat{a}_{ij},\hat{a}_{lm}) \tag{27}$$

These formulae approximate the bias and sensitivity of individual IO multipliers in the presence of correlations between the IO coefficients. Such correlations are prominent when there is information on row or column totals, and have not received adequate attention, according to Jackson and West (1989, p. 214).

To relate equations (26) and (27) to the literature, consider the case of independence. Then, $v(\hat{a}_{ij},\hat{a}_{ij})=\sigma_{ij}^2$ and $v(\hat{a}_{ij},\hat{a}_{lm})=0$ when $(i,j)\neq(l,m)$. This simplifies equations (26) and (27) further to

$$B(\hat{b}_{hk})\approx\sum_{i,j}b_{hi}b_{ji}b_{jk}\sigma_{ij}^2 \tag{28}$$

and

$$v(\hat{b}_{hk}\ \ \hat{b}_{rs})\approx\sum_{i,j}b_{hi}b_{jk}b_{ri}b_{js}\sigma_{ij}^2 \tag{29}$$

respectively. In particular, the own variances $((r, s) = (h, k))$ read as

$$v(\hat{b}_{hk}, \hat{b}_{hk}) \approx \sum_{i,j} (b_{hi} b_{jk} \sigma_{ij})^2 \tag{30}$$

The Leontief inverse is non-negative. This condition has been drawn from the so-called Hawkins–Simon conditions on the structural matrix **A** (Hawkins & Simon, 1949). In this case, the bias is non-negative, confirming the result of Simonovits (1975). Note that our result also holds when dependence is admitted. Note also that the bias can be signed as well as approximated, at least for small (co)variances.

The last assumption is made by West (1986). He studies column totals of multipliers, i.e. $\hat{M}_i = \Sigma_h \hat{b}_{hi}$, for independent technical coefficients. Summation over h yields

$$B(\hat{M}_k) \approx \sum_{i,j} M_i b_{ji} b_{jk} \sigma_{ij}^2 \tag{31}$$

and

$$v(\hat{M}_k, \hat{M}_k) \approx \sum_{i,j} (M_i b_{jk} \sigma_{ij})^2 \tag{32}$$

This coincides with the leading terms of West (1986). By the assumption of small variances, further factors in West (1986, 2.5 and 2.6), i.e. $(1 - 7b_{ji}^2\sigma_{ij}^2)^{-3/7} - 1$ for the bias and $[1 + (59/16)(b_{ji}\sigma_{ij})^2]^{128/59} - 1$ for the variance, are very small—in fact, they are less than 0.0005 and 0.0012 respectively. Therefore, they can be ignored. Although West's analysis is refuted and replaced by Kop Jansen (1998), West's formulae perform quite well in practice (ten Raa & Steel, 1994). Relative to the Monte Carlo results of ten Raa and Steel (1994, pp. 368–369), West's (1986) formulae are accurate for the means of the multipliers to the third decimal, and give standard errors and confidence intervals borders correct to the second decimal. The reason is that West's expressions essentially match first-order estimates of the bias and the variance. The non-negativity of the multipliers is not pertinent. Moreover, the assumption of independence can be dispensed with, as our preceding analysis shows. In fact, equations (26) and (27) resolve what Jackson and West (1987, p. 214) consider to be the most pressing issue: the analysis of coefficient interdependence.

6. Conclusions

Derived estimates are functions of direct model estimates. If the functions are non-linear, then derived estimates are biased. The bias and the variance have been approximated in this paper. Reduced-form estimates derived from structural form estimates, multiplier analysis and IO analysis are applications of increasing specificity for which corrections have been provided. Some thoughts on the estimation of substitution coefficients and of inverse demand elasticities follow.

In production theory, input demands are estimated directly and coefficients of substitution are evaluated *ex post.* Toevs (1982) has derived the variances of the coefficients of substitution for the trans-log function. Application of the present approximation theory yields the bias of the coefficient estimates (ten Raa, 1992).

In consumption theory, inversion plays a role in the stability of competitive adjustment processes (Sandberg, 1978) and the sustainability of natural monopolies (Baumol *et al.*, 1977). The structural form model is a system of demand equations, fulfilling weak gross substitutability and normality. Then, inverse demand exists and is decreasing (Sandberg, 1980). The elasticities of demand are non-positive, as is the bias. Apart from the sign, the situations of IO analysis and demand theory are similar and can be consolidated by the theory of *M*-functions (Kaper, 1991).

The sensitivity analysis enlightens the issue of important coefficients. For example, in IO analysis with independent coefficients, σ_{ij} is important for $\hat{b}_{hk}$ if $b_{hi}b_{jk}$ is large. In other words, if the subject of the analysis is the multiplier estimate b_{hk}, then the candidate for more precise estimation is indexed by (i,j), such that $b_{hi}b_{jk}$ is large.

If independence and non-negativity are maintained, then the bias can be shown to be larger than that in equation (28) (Kop Jansen, 1994). In this case, the mean of the Leontief inverse is obtained by the addition of the bias plus a further correction term to the Leontief inverse of the mean IO coefficients. The main relevance of the present paper, however, is that the bias and sensitivity formulae apply when structural coefficients are interdependent.

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