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FIXED POINTS OF COMPOSITIONS

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This note offers a condition for a composite correspondence to admit a fixed point. Applications in industrial organization are discussed.

1. Introduction

This note takes up the fixed point problem for composite correspondences. The problem stems from the theory of natural monopoly and oligopoly developed by Panzar and Willig (1977) and others. Here prices p elicit consumers demand quantities $q=Q(p)$ while the industry supplying q sets prices $P(q)=P[Q(p)]$ that render the supply of q invulnerable to the threat of competitive entry. As Sharkey (1981) observes, the question arises if the prices set agree with the ones which call forth the quantities demanded and supplied. In other words, does the composition of Q and P admit a fixed point?

To contribute to a positive answer, we wish to find classes S and F of sets and correspondences, respectively, such that the composition of $F_0: S_0 \to S_1, F_1: S_1 \to S_2, \ldots, F_n: S_n \to S_0$ ($S_0, \ldots, S_n$ of class S and $F_0, \ldots, F_n$ of class F) has a fixed point x_0 which belongs to $(F_n \circ \ldots \circ F_0)(x_0)$.

Section 2 will briefly discuss the mathematical history of the problem. Section 3 will present the main result: a condition for a composite correspondence to admit a fixed point. Applications in the aforementioned literature and industrial organization in general will be discussed in section 4.

2. Historical remarks

For $n=0$ we have one correspondence from a set to itself, the framework for the standard fixed point problem. Classical results were obtained by Kakutani (1941) for S, the class of convex Euclidian compacta and F, the

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class of upper semicontinuous convex-valued correspondences, and by Eilenberg and Montgomery (1946) who relaxed the convexity assumptions into contractibility ones.

For $n>0$, the subject of this note, the situation is complicated by the fact that class $\boldsymbol{F}$ need *not* be closed under the taking of compositions. For example, the composition of convex-valued correspondences is generally not convex-valued and therefore cannot be subjected to Kakutani's theorem. Nonetheless, the fixed point property itself is preserved when compositions are taken in the case of Kakutani's $\boldsymbol{S}$ and $\boldsymbol{F}$. This was proved in ten Raa (1981, 1984) by piecewise linear approximation of the first correspondence, F_0, induction on the number of correspondences or n, and a limiting argument. In that study, the generalization from convexity to contractibility, i.e., Eilenberg and Montgomery's $\boldsymbol{S}$ and $\boldsymbol{F}$, could only be *conjectured.* Then Debreu (1981) suggested an alternative proof, based on the selection of continuous functions f_i whose graphs are close to those of F_i and a limiting argument. Debreu was led to believe that the conjecture — if $\boldsymbol{S}$ and $\boldsymbol{F}$ are as in Eilenberg and Montgomery, then a composite correspondence has a fixed point — is not true. We shall return to this issue after the presentation of the main result.

3. Analysis

Our approach is as follows. The well-known case $n=0$ is taken for granted. Then a theorem shows how the fixed point property carries over to the general n case. Specific fixed point theorems for compositions can be obtained by combining the theorem with existing fixed point results, be they in a finite or infinite dimensional setting.

Definitions. A class is *closed* under the Cartesian product (or its permutation) if the Cartesian product (or its permutation) of any two members is also a member. For *correspondences* the Cartesian product $F\times G$ of $x\mapsto F(x)$ and $y\mapsto G(y)$ is defined by $(x,y)\mapsto F(x)\times G(y)$ and the permutation of the latter by $(x,y)\mapsto G(y)\times F(x)$.

Theorem. Let $\boldsymbol{S}$ and $\boldsymbol{F}$ be classes of sets and correspondences, respectively, such that any $F:S\to S$ (S of class $\boldsymbol{S}$ and F of class $\boldsymbol{F}$ has a fixed point. If $\boldsymbol{S}$ and $\boldsymbol{F}$ are closed under the Cartesian product and its permutation, then the composition of $F_0:S_0\to S_1$, $F_1:S_1\to S_2,\dots,$ $F_n:S_n\to S_0$ ($S_0,\dots,S_n$ of class $\boldsymbol{S}$ and $F_0,\dots,F_n$ of class $\boldsymbol{F}$) has a fixed point (even when it is not of class $\boldsymbol{F}$).

Proof. Let $S_0,\dots,S_n$ and $F_0,\dots,F_n$ be of classes $\boldsymbol{S}$ and $\boldsymbol{F}$ as in the theorem. Define $S=S_1\times\cdots\times S_n\times S_0$ and $F=F_1\times\cdots\times F_n$. By assumption, S is of class $\boldsymbol{S}$ and the permutation of $F\times F_0$ is of class $\boldsymbol{F}$. Since F goes from

$S_1 \times \cdots \times S_{n-1} \times S_n$ to $S_2 \times \cdots \times S_n \times S_0$ and F_0 from S_0 to S_1, the *permutation* of $F \times F_0$ goes from $(S_1 \times \cdots \times S_{n-1} \times S_n) \times S_0$ to $S_1 \times (S_2 \times \cdots \times S_n \times S_0)$. In other words, the permutation of $F \times F_0$ is a correspondence from S to S. Consequently, it has a fixed point $(x_1, \ldots, x_n, x_0) \in F_0(x_0) \times \cdots \times F_n(x_n)$. Take the last component of the established relationship and substitute the next to last through the first components. The result is $x_0 \in F_n(F_{n-1}(\ldots(F_0(x_0))\ldots)) = (F_n \circ \cdots \circ F_0)(x_0)$. Q.E.D.

Examples. In Kakutani's case, S is the class of convex Euclidian compacta and F the class of upper semicontinuous convex-valued correspondences. Since the Cartesian product (and its permutation) of any two convex Euclidian compacta is a convex Euclidian compactum, S is closed under the Cartesian product (and its permutation). And since the Cartesian product (and its permutation) of any two upper semicontinuous convex-valued correspondences is an upper semicontinuous convex-valued correspondence, F is closed too in this sense. Thus we have verified the condition of the theorem and therefore that the fixed point property is preserved when compositions are taken in the case of Kakutani's S and F.

In Eilenberg and Montgomery's case, convexity is generalized to contractibility. Note however that the just developed argument carries through, for the Cartesian product (and its permutation) of any two contractible sets is contractible! Thus it is permitted to take compositions in Eilenberg and Montgomery's setting: the fixed point property itself is preserved. This observation settles the conjecture mentioned in section 2 in a positive way.

Discussion. The crux of the theorem is that the fixed point property is preserved when compositions are taken provided that the assumed properties carry over to Cartesian products (rather than the compositions themselves). This condition is mild yet easily verifiable as we have seen in the examples.

4. Applications

We conclude with some brief mention of applications of the theorem.

Ten Raa (1981) developed the Kakutani case for the establishment of an equilibrium industrial organization in which the number of firms, their output vectors, the prices and the cost of entry are such that the market is cleared and (further) profitable entry is precluded. The analysis extends to the Sharkey (1981) model of natural monopoly with, however, interdependent demand [ten Raa (1984)].

Another related area of application is public enterprise pricing. Here the allocation of the costs of a bundle of services is defined to be anonymously equitable if no consumers subsidies are implied and the bundle is actually demanded at the implicit prices. Faulhaber and Levinson (1981) were able to prove that anonymous equity is achievable by direct application of

Kakutani's theorem, provided that demand for the services is independent. If, instead, the composition theorem is used, then conditions of interdependent demand may prevail [ten Raa (1983)].

Last but not least, the theorem is of potential use in general equilibrium analysis. An interpretation of the existence proof of competitive equilibrium is as follows. In our theorem, take $n=1$, F_0 the demand correspondence, and F_1 the inverse supply correspondence which associates support prices with boundary points of the production set. Then a fixed point of the composition constitutes a competitive equilibrium on the assumptions of McKenzie (1981). The present formulation, however, suggests the existence of more general equilibria, and, in particular, for economies with price-making firms. For all we need is a rule (F_1) which associates prices with production quantities, be they support prices or not.

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