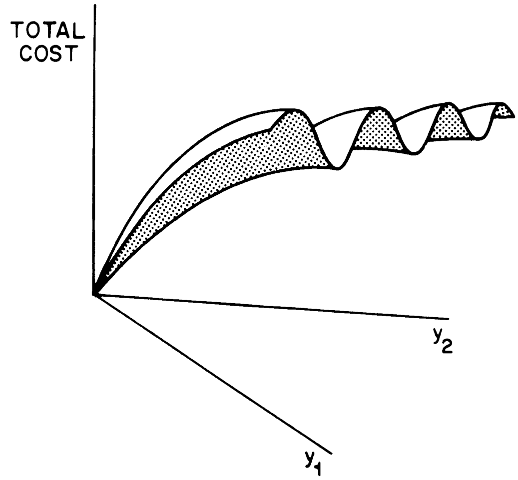

FIGURE 6

be noted that this is no extraordinary or pathological state of affairs. On the contrary, it is frequently found in multiproduct oligopoly industries in reality.[12]

We may note, finally, that though our natural monopoly criteria do not make for easy empirical testing, it is by no means unmanageable. Thus, as part of our research project on the cost of information supply, we have already found it possible with the aid of Proposition 12 to carry out tests of the hypothesis that there is subadditivity in the provision of a number of scientific journals by a single publisher (see the author and Braunstein).

APPENDIX

On the Cost-Minimizing Number of Firms When Ray Average Costs Decline

I conclude with two propositions which indicate how much (or how little) we know about the natural number of firms in an industry, given information only about cost behavior along each ray (economies of scale) but no information on transray cost behavior (economies of scope) (see also the author and Fischer).

PROPOSITION 13 (Thijs ten Raa): *If the cost function is strictly output-ray concave, then the optimal number of firms cannot exceed n, the number of commodities produced by the industry.*[13]

PROOF:

It is sufficient to prove that for every $(n + 1)$-tuple of output vectors $y^1, \ldots, y^{n+1}$ there exists a cheaper n-tuple of output vectors with the same total output. Because in n-dimensional space any $n + 1$ vectors are linearly dependent, $y^1, \ldots, y^{n+1}$ must be linearly dependent. Hence, one of them, say, without loss generality y^{n+1} is a linear combination of the others:

$$y^{n+1} = \sum_{i=1}^{n} C_i y^i, \quad C_i \geq 0, \quad \text{not all zero}$$

Let $$\lambda \epsilon \left[0, \min_{i=1,\ldots,n} \left(1 + \frac{1}{C_i}\right)\right]$$

then

$$\forall i \epsilon \{1, \ldots, n\}: (1 + C_i - \lambda C_i) y^i, \quad \lambda y^{n+1} \epsilon \overline{\mathcal{R}_n^+}$$

[12]It can be proved that if the cost function is ray concave everywhere but not subadditive, then the number of firms that minimize the cost of producing the industry output vector will never exceed n, the number of different products supplied by the industry. This by itself means that in most multiproduct industries even where costs satisfy the ray concavity requirement (which is stronger than the conventional concept of scale economies) it is perfectly possible to have dozens of firms corresponding to the dozens of products supplied by the industry. Nor need these firms specialize in the production of single products. Rather, it may be optimal for them to operate along rays in the interior of product space, depending on the transray behavior of the cost surface, as is indicated in Figure 6 where it may obviously be more economical for two firms to operate at the low points of the scalloped cross sections than above the axes. Moreover, if, on the usual (but not quite accurate) criterion of scale economies, the cost function has declining ray average costs but is not concave everywhere, then the cost-minimizing number of firms can even exceed the number of products of the industry. Examples showing this, and other proofs underlying this discussion, were provided by Thijs ten Raa and Dietrich Fischer. These materials are presented in the Appendix (Propositions 13 and 14).

[13]The proposition holds even if the concavity is *not* strict, provided there is no degeneracy in the sense used in linear programming.

and the sum of these $n + 1$ vectors equals

$$\sum_{i=1}^{n} y^i + \sum_{i=1}^{n} (C_i - \lambda C_i)y^i + \lambda \sum_{i=1}^{n} C_i y^i = \sum_{i=1}^{n} y^i + \sum_{i=1}^{n} C_i y^i = \sum_{i=1}^{n+1} y^i$$

C is output ray concave, hence

$$\forall i \epsilon \{1, \ldots, n\}: \; C\{(1 + C_i - \lambda C_i)y^i\}$$

is a concave function of λ and so is $C(\lambda y^{n+1})$. Hence

$$\sum_{i=1}^{n} C\{(1 + C_i - \lambda C_i)y^i\} + C(\lambda y^{n+1})$$

is a concave function of λ. Hence its minimum occurs at one of the end points of λ's range, that is, at

$$\lambda = 0 \quad \text{or}\ \lambda = \min_{i=1,\ldots,n} \left(1 + \frac{1}{C_i}\right) = 1 + \frac{1}{C_j} \quad \text{for some} \quad j \epsilon \{1, \ldots, n\}$$

When $\lambda = 0$, then the $(n + 1)$th firm does not produce anything. When $\lambda = 1 + 1/C_j$, then the jth firm does not produce anything. In either case only n firms are left, producing the industry output more cheaply than when $\lambda = 1$ under which we have the original outputs $y^1, \ldots, y^{n+1}$. However,

PROPOSITION 14 (Raa and Fischer): *Declining ray average costs alone do not preclude the optimality of a number of firms larger than the number of products supplied by the industry.*

PROOF:

The following is an example in which it is optimal to have three firms producing two commodities when ray average costs are (not strictly) declining and there are no fixed costs.[14] Let

$$y^I = \left(1 + \frac{\sqrt{3}}{3}, 2\right)$$

and, to simplify the argument (letting us deal with only three rays in output space) let C be such that production is relatively cheap along rays involving three particular output bundles

$$(y_2 \equiv 0, y_1 \equiv 0 \quad \text{and} \quad y_2 = \sqrt{3}\, y_1)$$

but prohibitively expensive along all other rays. In addition, let the cost function satisfy

$$C\left(\frac{y_2\sqrt{3}}{3}, y_2\right) = 2y_2 \quad \text{for} \quad y_2 \geq 0$$

$$C(0, y_2) = \sqrt{3}\, y_2 \quad \text{for} \quad y_3 \geq 0$$

$$C(y_1, 0) = y_1 \quad \text{for} \quad y_1 \geq 1; \sqrt{3} - 1 < C\left(1 - \frac{\sqrt{3}}{3}, 0\right) < 1$$

$$C(y_1, 0) \quad \text{is linear for} \quad y_1 \epsilon \left[0, 1 - \frac{\sqrt{3}}{3}\right] \quad \text{and for} \quad y_1 \epsilon \left[1 - \frac{\sqrt{3}}{3}, 1\right]$$

Then y^I is produced more cheaply by

$$y^1 = (1, 0), y^2 = \left(\frac{\sqrt{3}}{3}, 1\right) \quad \text{and} \quad y^3 = (0, 1)$$

To show this, because of the linearity of C, it is sufficient to show that (y^1, y^2, y^3) is cheaper than

$$\left(\left(1 - \frac{\sqrt{3}}{3}, 0\right), \left(\frac{2\sqrt{3}}{3}, 2\right)\right)$$

and

$$\left(\left(1 + \frac{\sqrt{3}}{3}, 0\right), (0, 2)\right)$$

the two pairs of output vectors capable of producing y^I. Thus:

$$C(y^1) + C(y^2) + C(y^3) = 1 + 2 + \sqrt{3} = 3 + \sqrt{3}$$

$$C\left(1 - \frac{\sqrt{3}}{3}, 0\right) + C\left(\frac{2\sqrt{3}}{3}, 2\right) > \sqrt{3} - 1 + 4 = 3 + \sqrt{3}$$

$$C\left(1 + \frac{\sqrt{3}}{3}, 0\right) + C(0, 2) = 1 + \frac{\sqrt{3}}{3} + 2\sqrt{3} = 1 + \left(2\frac{1}{3}\right)\sqrt{3} > 3 + \sqrt{3}$$

[14]Note that the addition of any positive fixed cost will make the ray average costs decline *strictly*, without affecting the example.